Reasoning in the Defeasible Description Logic $\mathcal{EL}_\bot$ —Computing Standard Inferences under Rational and Relevant Semantics

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Abstract

Defeasible Description Logics (DDLs) extend Description Logics with defeasible concept inclusions. Reasoning in DDLs often employs rational closure according to the (propositional) KLM postulates. A well-known approach to lift this closure to DDLs is by so-called materialisation. Previously investigated algorithms for materialisation-based reasoning employ reductions to classical reasoning using all Boolean connectors. As a first result in this paper, we present a materialisation-based algorithm for the sub-Boolean DDL $\mathcal{EL}_\bot$, using a reduction to reasoning in classical $\mathcal{EL}_\bot$, rendering materialisation-based defeasible reasoning tractable.

The main contribution of this article is a kind of canonical model construction, which can be used to decide defeasible subsumption and instance queries in $\mathcal{EL}_\bot$ under rational and the stronger relevant entailment. Our so-called typicality models can reproduce the entailments obtained from materialisation-based rational and relevant closure and, more importantly, obtain stronger versions of rational and relevant entailment. These do not suffer from neglecting defeasible information for concepts appearing nested inside quantifications, which all materialisation-based approaches do. We also show the computational complexity of defeasible subsumption and instance checking in our stronger rational and relevant semantics.

Keywords: Description Logics, Defeasible Reasoning, Subsumption, Instance checking

1. Introduction

Description Logics (DLs) are a collection of logics that have formally defined syntax and semantics. Most DLs are fragments of (the two-variable fragment of) First Order Logic. In DLs *concepts* describe groups of objects by means of other concepts (unary FOL predicates) and roles (binary relations). Such concepts can be assigned names or related to other concepts as sub- and super-concepts in the so-called TBox. Technically, the TBox can be viewed as a theory constraining the interpretation of the concepts. An important standard reasoning problem in DLs is to compute subsumption relationships between concepts. Such a relationship holds between two concepts, if all instances of one concept must necessarily be instances of the other (w.r.t. the TBox). Information on concrete objects is stored in the so-called ABox, where objects can be related to each other by roles or can be asserted as an instance of concepts, referring to the concepts defined in the TBox. Here a standard reasoning task is instance checking, i.e. deciding whether an individual does necessarily belong to a concept.

While classical DLs allow only for monotonic reasoning, there have been many approaches to extend DLs to non-monotonic reasoning. Earlier combinations of DLs and non-monotonic reasoning included defaults [1] or autoepistemic logic [2]. Recently investigated approaches are adaptations of circumscription to DLs [3] and

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defeasible DLs [4–9]. The latter have attracted a lot of research interest over the last years. Most defeasible DLs allow to state relationships between concepts that can be overwritten by more specific information and that characterise typical instances of a concept. Here, concepts may not always be consistent with all the defeasible information, hence some of it is defeated (disregarded) in order to perform consistent reasoning. There is a close relationship between the notion of typicality and defeasibility: the more defeasible information can be used for reasoning about a concept, the more typical it is considered.

Usually, the semantics of defeasible DLs are based on a translation of propositional rational reasoning introduced by Kraus, Lehmann and Magidor (KLM) in [10] to DLs. Recent investigations are on a typicality operator under preferential model semantics in [9], a syntactic materialisation based approach [4, 8], characterised with a different kind of preferential model semantics in [7] and extensions of rational reasoning to the stronger lexicographic and relevant closure in [6, 8].

The idea for lifting preferential and rational entailment from propositional logics [10] to description logics in [4] is to supply material implications based on defeasible axioms to the current subsumption or instance query. A known short-coming of this approach is that defeasible information is then only used locally for instances of the concepts mentioned in the query, but not necessarily for objects related to them by roles. Thus not all of the available and un-defeated information is used for the computation of consequences. To remedy this impairment was the main motivation for our investigations in [11, 12]. There, we considered subsumption in the DDL $\mathcal{EL}_\bot$ first based on rational closure [12] and then on relevant closure [11]. We focussed on the DL $\mathcal{EL}$, since it enjoys good computational properties: subsumption and instance checking can be computed in polynomial time [13, 14]. In this DL complex concepts are built by conjunctions and existential restrictions, which are a form of quantification and clearly not expressible by propositional logic. Despite its moderate expressivity, many applications rely on $\mathcal{EL}$, predominantly the Bio-Medical domain and Semantic Web applications using the web ontology language and its OWL 2 EL profile. In contrast to $\mathcal{EL}$, its extension $\mathcal{EL}_\bot$ can express disjointness of concepts and thus conflicting information. This ability renders satisfiability testing in the defeasible variant of $\mathcal{EL}_\bot$ non-trivial. We consider in this paper non-monotonic subsumption and non-monotonic instance checking in defeasible $\mathcal{EL}_\bot$. The DLs $\mathcal{EL}$ and $\mathcal{EL}_\bot$ have the canonical model property, i.e., such models can be embedded into all other models. Reasoning in logics that admit canonical models then boils down to computing the canonical models. In these models syntactical subconcepts from the TBox and, in case of ABox reasoning also the individuals are each represented by a domain element from the interpretation domain. Once the canonical model has been computed, the information for deciding subsumption or instance relationships can directly be read-off from this model.

Earlier approaches for computing these two inferences for defeasible DLs use reductions to classical reasoning with material implications [4, 8]. Generally, such reductions are desirable, since they allow to employ highly optimized DL reasoners for implementations instead of developing new systems from scratch. However, in order to remedy the impairment of materialisation, we do not concentrate on materialisation only, but we develop in this article a new kind of canonical models for the DDL $\mathcal{EL}_\bot$. This kind of models uses multiple representatives for each concept and thereby represents variations of the same concept satisfying different levels of typicality, i.e. different amounts of defeasible statements. This new kind of canonical models is called *typicality models.*

We investigate in this article reasoning by those typicality models for a range of different semantics which are determined by two parameters. One of them is the coverage of objects. Materialisation-based approaches have a propositional coverage of objects, in the sense that defeasible information is not propagated along role-relationships to other objects. Alternatively, defeasible information can (and should) be used for concepts appearing nested in existential restrictions or for objects that are related to the object of interest in an instance check. We extend typicality models to accommodate defeasible information for nested coverage. The other parameter determning the semantics is the strength of the conclusion. This parameter is in line with [8], presenting relevant closure, a *stronger* form of defeasible entailment than rational closure. We consider all of the semantics determined by the two parameters with two values each:

- *strength of defeasible conclusions*: relevant or rational
- *coverage of objects with defeasible information*: propositional or nested.

We investigate in this paper the reasoning services of defeasible subsumption and defeasible instance checking under all four combinations of these values. We begin with propositional coverage. The typicality models that admit propositional coverage, have only role-successors of minimal typicality in common. The canonical model for this class of models is therefore called the *minimal typicality* model. In order to be able to prove that typicality models supply those (defeasible) inferences that materialisation can provide and that it is not just an orthogonal approach, we first investigate reasoning by typicality models that provide propositional coverage and show that these can capture materialisation-based reasoning. In a second step, we extend typicality models to accommodate nested coverage. Typicality models that admit nested coverage of defeasible information can have many role-successors of maximised typicality in common. This means that role successors fulfil as much of the defeasible information as possible, as long as consistency is retained. These models are called *maximal typicality* models. We show that reasoning with maximal typicality models provides not only (strictly) more defeasible consequences, but also successfully allows defeasible consequences to be derived for quantified concepts and assertionally related objects.

The reasoning methods for deciding defeasible subsumption under propositional and under nested coverage have already been investigated by us in [11, 12]. The results on instance checking are new contributions in this article. Our methods on this reasoning task also alleviate the restrictions that earlier approaches impose on the knowledge base. In [4] the authors give a materialisation-based procedure for defeasible instance checking under rational closure only for unfoldable TBoxes, i.e. TBoxes that do not admit cyclic definitions and for ABoxes that fulfil certain syntactic conditions. Our algorithms for deciding the instance checking problem do not impose these restrictions. Furthermore, to the best of our knowledge we present the first reasoning algorithm for defeasible instance checking under relevant strength for propositional and nested coverage.

We investigate the computational complexity of defeasible subsumption and instance checking for the four different semantics. We obtain completeness results for rational strength and containment results for relevant strength.

Before we define and study reasoning by typicality models in this paper, we revisit the materialisation-based approach originally devised for the propositionally complete DL $\mathcal{ALC}$, which extends $\mathcal{EL}$. As materialisation is a reduction to classical reasoning and thus certainly interesting for practical applications of defeasible reasoning, we investigate the special case of materialisation in $\mathcal{EL}_\bot$. Materialisation for defeasible $\mathcal{ALC}$ uses all Boolean connectors to express material implications and employs EXP–Time complete $\mathcal{ALC}$ reasoning. Now, for $\mathcal{EL}_\bot$ one would like to attain a reasoning algorithm that remains polynomial. Casini et al. have claimed [4] that, generally, defeasible reasoning in a DDL does not increase the computational complexity of reasoning compared to its classical counterpart. While this is not hard to see for DLs allowing all Boolean connectors, where reasoning is already exponential, this is not immediately clear for sub-Boolean DDLs such as $\mathcal{EL}_\bot$. We devise a variant of the materialisation-based approach that does use only the expressivity of $\mathcal{EL}_\bot$ and thus can be handled by reasoners for that DL. Our materialisation for $\mathcal{EL}_\bot$ gives evidence to the claim that complexity of reasoning need not increase when moving from classical to defeasible sub-Boolean DLs.

The paper is structured as follows. We introduce the basic notions of Description Logics and discuss operations on models to prepare the construction of maximal typicality models in Section 2. In Section 3 we recall and discuss the materialisation-based approach from [4, 8, 15]. We present new variants of materialisation for the two reasoning problems considered by use of $\mathcal{EL}_\bot$-syntax and reasoning. We describe typicality models in Section 4 and characterise a canonical model for propositional coverage semantics, providing the same consequences as materialisation-based reasoning. In Section 5 we proceed to extend the typicality models to account for ABox individuals and show soundness and completeness of our approach for instance checking. Finally we investigate the computational complexity of this novel approach in Section 6 with the result of a strict increase in complexity over classical reasoning. We end the article with concluding remarks and an outlook towards future work in Section 7.

2. Preliminaries

2.1. Preliminaries on DLs

We introduce the basic notions of the (defeasible) DLs $\mathcal{ALC}$ and its fragment $\mathcal{EL}_\bot$, as well as their inferences. Starting from two disjoint sets N_C of concept names and N_R of role names, complex concepts can be defined inductively. Let C and D be $\mathcal{ALC}$-concepts and $r \in N_R$, then *(complex)* $\mathcal{ALC}$*-concepts* are:

- *named concepts* A ($A \in N_C$),
- the *top-concept* $\top$,
- the *bottom-concept* $\bot$,
- *negations* $\neg C$,
- *conjunctions* $C \sqcap D$,
- *disjunctions* $C \sqcup D$,
- *existential restrictions* $\exists r.C$, and
- *value restrictions* $\forall r.C$.

The DL $\mathcal{EL}_\bot$ is a sub-Boolean fragment of $\mathcal{ALC}$ allowing the concept constructors conjunction and existential restriction, as well as the concepts $\top$ and $\bot$.

The semantics of concepts is given by means of interpretations. An *interpretation* $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ consists of an *interpretation domain* $\Delta^\mathcal{I}$ and a *mapping function* $\cdot^\mathcal{I}$ that assigns subsets of the domain $\Delta^\mathcal{I}$ to concept names and binary relations over the domain $\Delta^\mathcal{I}$ to role names. The top-concept is interpreted as the whole domain ($\top^\mathcal{I} = \Delta^\mathcal{I}$) and the bottom-concept as the empty set ($\bot^\mathcal{I} = \emptyset$). The complex concepts are interpreted as follows:

- $(\neg C)^\mathcal{I} = \Delta^\mathcal{I} \setminus C^\mathcal{I}$,
- $(C \sqcap D)^\mathcal{I} = C^\mathcal{I} \cap D^\mathcal{I}$,
- $(C \sqcup D)^\mathcal{I} = C^\mathcal{I} \cup D^\mathcal{I}$,
- $(\exists r.C)^\mathcal{I} = \{d \in \Delta^\mathcal{I} \mid \exists e.(d,e) \in r^\mathcal{I} \text{ and } e \in C^\mathcal{I}\}$, and
- $(\forall r.C)^\mathcal{I} = \{d \in \Delta^\mathcal{I} \mid (d,e) \in r^\mathcal{I} \implies e \in C^\mathcal{I}\}$.

If in an interpretation $\mathcal{I}$ $(d,e) \in r^\mathcal{I}$ holds, then e is called a *role successor* of d, conversely, d is called a *role predecessor* of e and (d,e) is called a connection (or edge) in r. For concepts C and role names r, $C^\mathcal{I}$ and $r^\mathcal{I}$ are frequently called the extensions of C and r under the interpretation $\mathcal{I}$. A concept C is *satisfied* by an interpretation $\mathcal{I}$ *iff* $C^\mathcal{I} \neq \emptyset$. The expression "$C$ is satisfied by $\mathcal{I}$ with $d \in \Delta^\mathcal{I}$", implies $d \in C^\mathcal{I}$.

DL ontologies can state (monotonous) relationships between concepts. Let C and D be concepts. A *general concept inclusion axiom* (GCI) is of the form: $C \sqsubseteq D$. A *TBox* $\mathcal{T}$ is a finite set of GCIs. A GCI $C \sqsubseteq D$ is *satisfied* in an interpretation $\mathcal{I}$, *iff* $C^\mathcal{I} \subseteq D^\mathcal{I}$ (written $\mathcal{I} \models C \sqsubseteq D$). An interpretation $\mathcal{I}$ is a *model of a TBox* $\mathcal{T}$, *iff* $\mathcal{I}$ satisfies all GCIs in $\mathcal{T}$ (written $\mathcal{I} \models \mathcal{T}$). The TBox is used to express conceptual knowledge, it poses as a set of constraints on interpretations. Sometimes, in addition to conceptual knowledge, one needs to express facts about named individuals (in the world), e.g. entries in a database. The set of individual names N_I is introduced as a disjoint set from N_C and N_R and we frequently use lower-case letters $a, b \in N_I$ for individuals. The semantics of individuals are given by interpretations, with $a^\mathcal{I} \in \Delta^\mathcal{I}$. In order to express facts about individuals, *concept assertion axioms* $C(a)$ and *role assertion axioms* $r(a,b)$ are collected in a finite set, called the ABox $\mathcal{A}$. An interpretation satisfies a concept assertion $C(a)$ *iff* $a^\mathcal{I} \in C^\mathcal{I}$ and a role assertion $r(a,b)$ *iff* $(a^\mathcal{I}, b^\mathcal{I}) \in r^\mathcal{I}$, and it is a model of the ABox $\mathcal{A}$, if it satisfies all the assertion axioms in

$\mathcal{A}$ (written $\mathcal{I} \models \mathcal{A}$). The combination of an ABox $\mathcal{A}$ and a TBox $\mathcal{T}$ is called a knowledge base $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ and $\mathcal{I} \models \mathcal{K}$ *iff* $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \models \mathcal{A}$.

Based on the notion of a model, DL reasoning problems are defined. A knowledge base $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ is called *consistent iff* there exists a model of $\mathcal{K}$. A concept is *consistent* w.r.t. a TBox $\mathcal{T}$ (or KB $\mathcal{K}$) *iff* some model of $\mathcal{T}$ (resp. $\mathcal{K}$) satisfies the concept. A concept C is *subsumed by* a concept D w.r.t. a TBox $\mathcal{T}$ (written $C \sqsubseteq_\mathcal{T} D$ or $\mathcal{T} \models C \sqsubseteq D$) *iff* $C^\mathcal{I} \subseteq D^\mathcal{I}$ holds in all models $\mathcal{I}$ of $\mathcal{T}$. Two TBoxes $\mathcal{T}_1$ and $\mathcal{T}_2$ are equivalent, *iff* $\mathcal{I} \models \mathcal{T}_1 \iff \mathcal{I} \models \mathcal{T}_2$ holds for all interpretations $\mathcal{I}$. A statement $C(a)$ is entailed by a knowledge base $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ (written $\mathcal{K} \models C(a)$) *iff* $a^\mathcal{I} \in C^\mathcal{I}$ for all models $\mathcal{I}$ of $\mathcal{K}$. *Instance checking* is to decide whether $\mathcal{K} \models C(a)$ holds for given C, a and $\mathcal{K}$.

Note that subsumption w.r.t. a consistent knowledge base is independent of the ABox i.e. $(\mathcal{A}, \mathcal{T}) \models C \sqsubseteq D$ *iff* $\mathcal{T} \models C \sqsubseteq D$. In contrast to this, instance checking depends on a (non-empty) TBox, e.g. $\mathcal{K} \models C(a)$ and $\mathcal{K} \models C \sqsubseteq D$ implies $\mathcal{K} \models D(a)$. Inconsistency of a knowledge base can also be captured with subsumption as $\mathcal{K} \models \top \sqsubseteq \bot$ or instance checks $\mathcal{K} \models \bot(a)$.

We fix some notation for the remainder of the paper to access parts of knowledge bases or concepts. Let X denote a concept or a TBox, ABox, KB, then $sig(X)$ denotes the *signature* of X, i.e. the set of concept, role and individual names occurring in X. We define $sig_C(X) = sig(X) \cap N_C$, $sig_R(X) = sig(X) \cap N_R$ and $sig_I(X) = sig(X) \cap N_I$ (the latter is mostly used with an ABox or KB X). We also define the set $Qc(X)$ of *quantified concepts* in X as $F \in Qc(X)$ *iff* $\exists r.F$ syntactically occurs in X for some $r \in N_R$.

In extensions of $\mathcal{EL}$ that are in the Horn fragment of DLs, canonical models are widely used for reasoning [13]. For an $\mathcal{EL}_\bot$-TBox $\mathcal{T}$, the *canonical model* $\mathcal{I}_\mathcal{T} = (\Delta^{\mathcal{I}_\mathcal{T}}, \cdot^{\mathcal{I}_\mathcal{T}})$ of $\mathcal{T}$ with $\Delta^{\mathcal{I}_\mathcal{T}} = \{d_F \mid F \in Qc(\mathcal{T})\}$ has the mapping function satisfying the conditions:

- $d_F \in A^{\mathcal{I}_\mathcal{T}}$ *iff* $F \sqsubseteq_\mathcal{T} A$ and
- $(d_F, d_G) \in r^{\mathcal{I}_\mathcal{T}}$ *iff* $F \sqsubseteq_\mathcal{T} \exists r.G$.

Once the canonical model is computed, subsumption relationships between concepts can be directly read-off from it [13, 14], i.e. $\mathcal{T} \models C \sqsubseteq D$ *iff* $d_C \in D^{\mathcal{I}_\mathcal{T}}$.

A canonical model $\mathcal{I}_{\mathcal{A},\mathcal{T}}$ can be constructed for a knowledge base $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ with $\Delta^{\mathcal{I}_{\mathcal{A},\mathcal{T}}} = \{d_F \mid F \in Qc(\mathcal{K})\} \cup \{d_a \mid a \in sig_I(\mathcal{A})\}$ such that the mapping function satisfies the same conditions as $\cdot^{\mathcal{I}_\mathcal{T}}$ as well as:

- $a^{\mathcal{I}_{\mathcal{A},\mathcal{T}}} = d_a$,
- $d_a \in A^{\mathcal{I}_{\mathcal{A},\mathcal{T}}}$ *iff* $(\mathcal{A}, \mathcal{T}) \models A(a)$,
- $(d_a, d_G) \in r^{\mathcal{I}_{\mathcal{A},\mathcal{T}}}$ *iff* $(\mathcal{A}, \mathcal{T}) \models (\exists r.G)(a)$ and
- $(d_a, d_b) \in r^{\mathcal{I}_{\mathcal{A},\mathcal{T}}}$ *iff* $r(a, b) \in \mathcal{A}$.

Entailments of $\mathcal{K}$, including instance relationships, can be read-off from $\mathcal{I}_{\mathcal{A},\mathcal{T}}$, i.e. $\mathcal{K} \models C(a)$ *iff* $d_a \in C^{\mathcal{I}_{\mathcal{A},\mathcal{T}}}$.

2.1.1. Defeasible Description Logics

In defeasible DLs it can be stated that a concept is subsumed by another concept as long as there is no contradicting information. A *defeasible concept inclusion* (DCI) is of the form $C \sqsubsim D$ and states that elements of C are *usually* also elements of D. A *DBox* $\mathcal{D}$ is a finite set of DCIs. A *defeasible knowledge base* (DKB) $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ consists of an ABox $\mathcal{A}$, a TBox $\mathcal{T}$ and a DBox $\mathcal{D}$. When convenient, we skip a component of the triple and assume that it is the empty set. The definitions for $sig(X)$, $sig_C(X)$, $sig_R(X)$ and $Qc(X)$ extend to DBoxes or DKBs in the obvious way.

The semantics of DBoxes differ from the ones for TBoxes, since DCIs need not hold at each element in the model whereas GCIs do. The satisfaction of a finite set of DCIs $\mathcal{D}$ for $d \in \Delta^\mathcal{I}$ is captured by $\mathcal{I}, d \models \mathcal{D}$ *iff* $\forall G \sqsubsim H \in \mathcal{D}. d \in G^\mathcal{I} \implies d \in H^\mathcal{I}$. Intuitively, the more DCIs are satisfied by a domain element d, the more typical d can be considered. Usually, the semantics of DBoxes is given by means of ranked/ordered interpretations—called preferential model semantics [7, 9]. Instead of using these, we define a new kind of model for DKBs (in Sect. 4) that extends canonical models for $\mathcal{EL}_\bot$. The main idea is to use several copies of the representatives, such as d_F, for each existentially quantified concept, where each copy satisfies a different set of DCIs from the powerset of the DBox. We want to develop a decision procedure for (defeasible) subsumption relationships between concepts, say C and D, and (defeasible) instance checks, say $C(a)$, w.r.t. a given DKB $\mathcal{K}$ under both, rational and relevant semantics.

For the remainder of the article we make three simplifying assumptions about knowledge bases for the sake of ease of presentation. We assume w.l.o.g. that

1. concepts C and D appear syntactically in $Qc(\mathcal{K})$ which can be achieved by adding $\exists r.E \sqsubseteq \top$ with $E \in \{C, D\}$ to $\mathcal{T}$,
2. all quantified concepts in $\mathcal{K}$ are consistent i.e. , $\forall F \in Qc(\mathcal{K}).F \not\sqsubseteq_{\mathcal{T}} \bot$ and thus $\bot \notin Qc(\mathcal{K})$, and
3. every ABox $\mathcal{A}$ is conjunction-free, i.e. $(C \sqcap D)(a) \notin \mathcal{A}$ for any two concepts C and D (for every ABox there exists an equivalent conjunction-free ABox).

Assumptions 1 and 3 obviously preserve generality. Assumption 2 preserves generality, because every KB can be transformed to a KB where all quantified concepts are consistent by replacing entire existential restrictions $\exists r.X$ by $\bot$, whenever $X \sqsubseteq_{\mathcal{T}} \bot$. Clearly the resulting KB is has the same models as the original one, since $X \equiv_{\mathcal{T}} \bot \implies \exists r.X \equiv_{\mathcal{T}} \bot$.

2.2. Basic Operations on Interpretations

We introduce operations on interpretations that we need in the technical constructions later on. The first operation, (cross-)product of interpretations, is a standard notion in classical description logics that is commonly used in proofs.

Definition 2.1. Given two interpretations $\mathcal{I}$ and $\mathcal{J}$. The *product interpretation* of $\mathcal{I}$ and $\mathcal{J}$ is defined as $\mathcal{I} \times \mathcal{J} = (\Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}, \cdot^{\mathcal{I}\times\mathcal{J}})$, where

- $A^{\mathcal{I}\times\mathcal{J}} = A^{\mathcal{I}} \times A^{\mathcal{J}}$ $(A \in N_C)$
- $r^{\mathcal{I}\times\mathcal{J}} = \{((a, b), (c, d)) \mid (a, c) \in r^{\mathcal{I}} \land (b, d) \in r^{\mathcal{J}}\}$ $(r \in N_R)$.
- $a^{\mathcal{I}\times\mathcal{J}} = (a^{\mathcal{I}}, a^{\mathcal{J}})$

In the description logic $\mathcal{EL}_{\bot}$, the set of all models of a knowledge base is closed under products [16, 17]. This often helps us to combine two models that each *do not* support a certain entailment, e.g. $d \notin C^{\mathcal{I}}$ and $e \notin D^{\mathcal{J}}$, in order to obtain a single model not supporting either of the respective entailments, e.g. $d \notin C^{\mathcal{I}\times\mathcal{J}}$ and $e \notin D^{\mathcal{I}\times\mathcal{J}}$.

The next notion, lifting set operations to interpretations, is less commonly used in DLs, as some set operations over interpretations are only useful in very specific cases.

Definition 2.2. For two interpretations $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}), \mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ and an operation $\bowtie \in \{\cap, \cup\}$, we define

- $\mathcal{I} \bowtie \mathcal{J} = (\Delta^{\mathcal{I}} \bowtie \Delta^{\mathcal{J}}, \cdot^{\mathcal{I}\bowtie\mathcal{J}})$ with $A^{\mathcal{I}\bowtie\mathcal{J}} = A^{\mathcal{I}} \bowtie A^{\mathcal{J}}$ (for $A \in N_C$) and $r^{\mathcal{I}\bowtie\mathcal{J}} = r^{\mathcal{I}} \bowtie r^{\mathcal{J}}$ (for $r \in N_R$)
- $\mathcal{I} \subseteq \mathcal{J}$ *iff* $\forall A \in N_C.A^{\mathcal{I}} \subseteq A^{\mathcal{J}}$, $\forall r \in N_R.r^{\mathcal{I}} \subseteq r^{\mathcal{J}}$ and $\forall a \in N_I.a^{\mathcal{I}} = a^{\mathcal{J}}$. $\mathcal{I}$ is called a *sub-interpretation* of $\mathcal{J}$.

Remark 2.3. In this article, we will use the notion of interpretation intersection and sub-interpretations exclusively on two interpretations over the same domain. The notion of sub-interpretations adapts to strict inclusion and equality in the obvious way. The intersection of two interpretations over a shared domain clearly yields the same domain again, the same holds for the union of such interpretations. Eventually, the union of interpretations is used both, for interpretations over a shared domain as well as interpretations over arbitrary domains. In case individuals $a \in sig_I(\mathcal{K})$ are considered, in this article, either only one of the involved interpretations considers extensions of the individuals, therefore it is clear how to define $a^{\mathcal{I}\bowtie\mathcal{J}}$, or every individual is mapped to the same domain element under both interpretations, in which case there is also only one sensible definition for $a^{\mathcal{I}\bowtie\mathcal{J}}$.

Sometimes we need to manipulate interpretations and would like to specify the resulting interpretation only by the change of the interpretation mappings. This can be done by means of substitutions.

Definition 2.4. Let $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$ be an interpretation. We call substitutions of the kind

- A/Y (with $Y \subseteq \Delta$, $A \in N_C$) *concept substitutions*, and
- r/X (with $X \subseteq \Delta \times \Delta$, $r \in N_R$) *role substitutions*,
- a/d (with $d \in \Delta$, $a \in N_I$) *individual substitutions.*

Let σ be a finite set of role, concept and individual substitutions, where each concept, role or individual name is substituted at most once. The interpretation $\mathcal{I}[\sigma] = (\Delta, \cdot^{\mathcal{I}[\sigma]})$ is obtained from $\mathcal{I}$ by applying the substitutions in σ to $\mathcal{I}$, where

- $A^{\mathcal{I}[\sigma]} = \begin{cases} Y & \text{, if } A/Y \in \sigma \\ A^{\mathcal{I}} & \text{, otherwise} \end{cases}$ (for $A \in N_C$)
- $r^{\mathcal{I}[\sigma]} = \begin{cases} X & \text{, if } r/X \in \sigma \\ r^{\mathcal{I}} & \text{, otherwise} \end{cases}$ (for $r \in N_R$)
- $a^{\mathcal{I}[\sigma]} = \begin{cases} d & \text{, if } a/d \in \sigma \\ a^{\mathcal{I}} & \text{, otherwise} \end{cases}$ (for $a \in N_I$)

Applying substitutions to interpretations is useful, in particular, because it allows to easily reduce (e.g. $\mathcal{I}[A/\emptyset, B/\{d\}]$) or extend (e.g. $\mathcal{I}[r/r^{\mathcal{I}} \cup \{(d, e)\}]$) a given interpretation $\mathcal{I}$, in a succinct and well-defined way by specifying only the local changes.

A standard notion in DLs is disjointness of interpretations. Two interpretations are disjoint if they have disjoint domains. An important property for $\mathcal{EL}_\bot$ is that the disjoint union of two models of a TBox is again a model of that TBox. We generalise disjointness of two interpretations by allowing both domains to share some elements. If certain conditions are met, we can still obtain a property that normally holds for disjoint interpretations.

Definition 2.5. An interpretation $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ is *quasi-disjoint* from an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *iff*

- $\forall A \in N_C$ holds $A^{\mathcal{J}} \cap \Delta^{\mathcal{I}} = \emptyset$ and
- $\forall r \in N_R$ holds $r^{\mathcal{J}} \cap \big(\Delta^{\mathcal{I}} \times (\Delta^{\mathcal{I}} \cup \Delta^{\mathcal{J}})\big) = \emptyset$.

Quasi-disjointness relaxes disjointness in the sense that the shared domain elements do not occur in the range of interpretation mappings for concepts or roles of $\mathcal{J}$, i.e. $\mathcal{J}$ "attaches no concept or role information" to shared elements. Thus quasi-disjointness is not a symmetric relation, however it properly generalises disjointness, as $\mathcal{I}$ is quasi-disjoint from $\mathcal{J}$ (and the other way around), if $\Delta^{\mathcal{I}}$ is disjoint from $\Delta^{\mathcal{J}}$. Disjoint unions of interpretations trivially satisfy $C^{\mathcal{I} \uplus \mathcal{J}} \cap \Delta^{\mathcal{I}} = C^{\mathcal{I}}$. That means, one can effectively reconstruct the original interpretation mapping of $\mathcal{I}$ from the disjoint union $\mathcal{I} \uplus \mathcal{J}$. This property holds already for (non-disjoint) unions of two interpretations, where one interpretation is quasi-disjoint from the other.

Proposition 2.6. *For two interpretations $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ and $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ s.t. $\mathcal{J}$ is quasi-disjoint from $\mathcal{I}$ it holds that*

$$C^{\mathcal{I} \cup \mathcal{J}} \cap \Delta^{\mathcal{I}} = C^{\mathcal{I}}$$

for all $\mathcal{EL}_\bot$ concepts C.

PROOF. We prove the claim by induction on the concept C. The case of $C = \bot$ is trivial. In case $C = A$ with $A \in N_C$, $A^{\mathcal{J}} \cap \Delta^{\mathcal{I}} = \emptyset$ holds by quasi-disjointness of $\mathcal{J}$ from $\mathcal{I}$ and thus $A^{\mathcal{I}\cup\mathcal{J}} \cap \Delta^{\mathcal{I}} = (A^{\mathcal{I}} \cup A^{\mathcal{J}}) \cap \Delta^{\mathcal{I}} = A^{\mathcal{I}}$. Assume as induction hypothesis (IH) the claim holds for the $\mathcal{EL}_\bot$ concepts E, F. For $C = E \sqcap F$ it holds that

$$\begin{aligned}
(E \sqcap F)^{\mathcal{I}\cup\mathcal{J}} \cap \Delta^{\mathcal{I}} &= E^{\mathcal{I}\cup\mathcal{J}} \cap F^{\mathcal{I}\cup\mathcal{J}} \cap \Delta^{\mathcal{I}} \\
&= (E^{\mathcal{I}\cup\mathcal{J}} \cap \Delta^{\mathcal{I}}) \cap (F^{\mathcal{I}\cup\mathcal{J}} \cap \Delta^{\mathcal{I}}) \\
&\overset{IH}{=} E^{\mathcal{I}} \cap F^{\mathcal{I}} \\
&= (E \sqcap F)^{\mathcal{I}}.
\end{aligned}$$

For the case $C = \exists r.E$, it holds that $r^{\mathcal{J}} \cap \big(\Delta^{\mathcal{I}} \times (\Delta^{\mathcal{I}} \cup \Delta^{\mathcal{J}})\big) = \emptyset$ by quasi-disjointness of $\mathcal{J}$ from $\mathcal{I}$, which implies $r^{\mathcal{I}\cup\mathcal{J}} \cap (\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}\cup\mathcal{J}}) = r^{\mathcal{I}}$ $(*)$. Clearly,

$$(\exists r.E)^{\mathcal{I}\cup\mathcal{J}} \cap \Delta^{\mathcal{I}} = \{d \in \Delta^{\mathcal{I}} \mid \exists e \in \Delta^{\mathcal{I}\cup\mathcal{J}}.(d,e) \in r^{\mathcal{I}\cup\mathcal{J}} \wedge e \in E^{\mathcal{I}\cup\mathcal{J}}\}.$$

The pairs (d, e) considered in the condition of this set belong to $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}\cup\mathcal{J}}$ (since $d \in \Delta^{\mathcal{I}}$) and therefore, $(*)$ allows to conclude $(d, e) \in r^{\mathcal{I}}$. If $(d, e) \in r^{\mathcal{I}}$, then $e \in \Delta^{\mathcal{I}}$ and thus $e \in E^{\mathcal{I}\cup\mathcal{J}} \cap \Delta^{\mathcal{I}} = E^{\mathcal{I}}$ by the induction hypothesis. It follows that

$$\begin{aligned}
&\{d \in \Delta^{\mathcal{I}} \mid \exists e \in \Delta^{\mathcal{I}\cup\mathcal{J}}.(d,e) \in r^{\mathcal{I}\cup\mathcal{J}} \wedge e \in E^{\mathcal{I}\cup\mathcal{J}}\} \\
&= \{d \in \Delta^{\mathcal{I}} \mid \exists e \in \Delta^{\mathcal{I}}.(d,e) \in r^{\mathcal{I}} \wedge e \in E^{\mathcal{I}}\} \\
&= (\exists r.E)^{\mathcal{I}}.
\end{aligned}$$

□

Proposition 2.6 allows to characterise certain properties of some interpretation $\mathcal{I}$, then introduce an interpretation $\mathcal{J}$ that is quasi-disjoint from $\mathcal{I}$ and carry those properties over to $\mathcal{I}\cup\mathcal{J}$.

To motivate our approach for reasoning under rational and relevant semantics in defeasible $\mathcal{EL}_\bot$, we recall first earlier approaches for this task and discuss several effects that are inherent for the characterised defeasible entailment relations.

3. On Materialisation-based Approaches for Reasoning in DDLs

In this section we introduce and discuss reasoning algorithms for DDLs that use a reduction to classical reasoning to determine (KLM-style) rational and relevant non-monotonic inferences. The reduction to classical reasoning is obtained by the materialisation of DCIs. What we call the *materialisation-based approach* was initially introduced in [4] for defeasible subsumption and instance checking under rational closure. Deciding defeasible subsumption was lifted to relevant closure (along with a more sophisticated version for subsumption under rational closure) in [8]. For subsumption under both closures, we refer to [8], and for instance checking under rational closure, we refer to [4], as it has not been introduced for relevant closure prior to this article.

The materialisation-based approach proceeds in two steps. First, for a given query, a consistent subset of the DBox is determined, depending on the chosen strength of the defeasible conclusion (rational, relevant). This set of DCIs is then materialised, in order to "enrich" the query with defeasible information. For subsumption, this enrichment occurs syntactically on the left-hand side of the subsumption query, i.e. the considered domain elements are restricted to those satisfying the materialised part of the consistent DBox subset. For instance checking, all individuals in the ABox are enriched by the materialisation of a consistent DBox subset (depending on the individual), using concept assertion axioms to extend the ABox.

Since the formalism introduced as the main contribution of this article relies on the use of the DL $\mathcal{EL}_\bot$ we investigate the materialisation-based approach within $\mathcal{EL}_\bot$, which is not a trivial matter, as the main aspect of materialising DCIs ($C \mathrel{\underset{\sim}{\sqsubset}} D \rightsquigarrow \neg C \sqcup D$) cannot be done while remaining in $\mathcal{EL}_\bot$. As the first result of this article, the consequence of Section 3.2 is that the materialisation-based approach remains polynomial for the DL $\mathcal{EL}_\bot$ as claimed in [4]. This result is not clear from the approach described in [4, 8].

3.1. Revisiting Materialisation-based Defeasible Reasoning

We briefly recall the reduction algorithms for reasoning about defeasible KBs. More precisely, we discuss subsumption under rational and relevant closure as described by Casini et al. in [8], as well as the reduction algorithm for instance query answering under rational closure as introduced in [4]. For technical details and in-depth motivation, we refer the reader to the papers [4, 8].

Partitioning of the DBox. To decide defeasible subsumption $C \mathrel{\underset{\sim}{\sqsubset}} D$ w.r.t. a given DKB $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ under rational closure the algorithm in [8] uses materialisation of DCIs. The idea is, that an element that belongs to $\neg E \sqcup F$ also satisfies the DCI $E \mathrel{\underset{\sim}{\sqsubset}} F$. A concept C can be *enriched* with defeasible information, for instance by considering classical consequences of the concept $C \sqcap (\neg E \sqcup F)$, i.e. consequences that follow from those elements in C that also satisfy $E \mathrel{\underset{\sim}{\sqsubset}} F$. To that end, the materialisation of a set of DCIs $\mathcal{D}$ is defined as $\overline{\mathcal{D}} = \bigsqcap_{E \mathrel{\underset{\sim}{\sqsubset}} F \in \mathcal{D}} (\neg E \sqcup F)$. Since C might be inconsistent w.r.t. the materialisation of the entire DBox $\mathcal{D}$, the algorithm needs to determine a subset $\mathcal{D}' \subseteq \mathcal{D}$ whose materialisation is consistent with C and $\mathcal{T}$ in order to decide whether $\overline{\mathcal{D}'} \sqcap C \sqsubseteq_{\mathcal{T}} D$ holds. To obtain $\mathcal{D}'$, $\mathcal{D}$ is iteratively reduced to that subset containing all DCIs whose left-hand sides are inconsistent in conjunction with the materialisation of the current DBox:

$$\mathcal{E}(\mathcal{D}) = \{C \mathrel{\underset{\sim}{\sqsubset}} D \in \mathcal{D} \mid \mathcal{T} \models \overline{\mathcal{D}} \sqcap C \sqsubseteq \bot\}.$$

They define $\mathcal{E}^1(\mathcal{D}) = \mathcal{E}(\mathcal{D})$ and $\mathcal{E}^j(\mathcal{D}) = \mathcal{E}(\mathcal{E}^{j-1}(\mathcal{D}))$ (for $j > 1$). Using $\mathcal{E}()$, the DCIs in $\mathcal{D}$ can be ranked according to their level of exceptionality, i.e. , $r_{\mathcal{K}}(G \mathrel{\underset{\sim}{\sqsubset}} H) = i - 1$, for the smallest i s.t. $G \mathrel{\underset{\sim}{\sqsubset}} H \notin \mathcal{E}^i(\mathcal{D})$, or $r_{\mathcal{K}}(G \mathrel{\underset{\sim}{\sqsubset}} H) = \infty$ if no such i exists. A DKB $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ is *well-separated* if no DCI in $\mathcal{D}$ has an infinite rank of exceptionality [7]. Since every DKB (that includes the concept $\bot$) can be transformed into a well-separated one deciding a polynomial number of classical subsumptions, we assume w.l.o.g. that all DKBs in this article are well-separated. Based on the level of exceptionality $r_{\mathcal{K}}()$, the algorithm from [8] partitions the DBox $\mathcal{D}$ into $(E_0, E_1, \ldots, E_n)$ where $E_i = \{G \mathrel{\underset{\sim}{\sqsubset}} H \in \mathcal{D} \mid r_{\mathcal{K}}(G \mathrel{\underset{\sim}{\sqsubset}} H) = i\}$, i.e. $\mathcal{D} = \bigcup_{i=0}^{n} E_i$. To find the maximal (w.r.t. cardinality) subset $\mathcal{D}'$ of $\mathcal{D}$, whose materialisation is consistent with C and $\mathcal{T}$ the procedure starts with $\mathcal{D}' = \mathcal{D}$. If $\overline{\mathcal{D}'} \sqcap C \sqsubseteq_{\mathcal{T}} \bot$, then E_i is removed from $\mathcal{D}'$ for the smallest not yet used i. This test and removal is done iteratively until a subset of $\mathcal{D}$ is reached whose materialisation is consistent with C and $\mathcal{T}$. We denote the resulting subsets of $\mathcal{D}$ by

$$\mathcal{D}_i = \bigcup_{j=i}^{n} E_j,$$

where $0 \leq i \leq n$ is the number of iterations that the above procedure was executed. We frequently use the expression "consider the least i for which $\mathcal{D}_i$ is consistent with C".

While rational closure treats inconsistencies with the granularity of the partitions E_i, relevant closure uses a more fine-grained treatment. To illustrate this, let $G \mathrel{\underset{\sim}{\sqsubset}} H \in E_0$ and assume that C is only consistent with $\mathcal{D} \setminus E_0$ (or its subsets). In this situation

$$(\neg G \sqcup H) \sqcap \overline{\mathcal{D} \setminus E_0} \sqcap C \ \sqsubseteq_{\mathcal{T}} \ \bot$$

need not hold, since the inconsistency may be due to other DCIs in E_0. Still $G \mathrel{\underset{\sim}{\sqsubset}} H$ is never used for reasoning about C. This effect is called *inheritance blocking*, as it might be possible to include $G \mathrel{\underset{\sim}{\sqsubset}} H$ for reasoning about C, but other DCIs induce some inconsistency and so block the inheritance of property $G \mathrel{\underset{\sim}{\sqsubset}} H$ for C. Under relevant closure, only DCIs that are *relevant* for the inconsistency of C are disregarded, thereby averting inheritance blocking. General relevant closure and two specific constructions (basic and minimal relevant closure) are introduced in [8] in terms of justification.

Definition 3.1. Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB, $\mathcal{J} \subseteq \mathcal{D}$, and C a concept. $\mathcal{J}$ is a *C-justification* w.r.t. $\mathcal{K}$ *iff* $\overline{\mathcal{J}} \sqcap C \sqsubseteq_{\mathcal{T}} \bot$ and $\overline{\mathcal{J}'} \sqcap C \not\sqsubseteq_{\mathcal{T}} \bot$ for all $\mathcal{J}' \subset \mathcal{J}$.

Let *justifications*$(\mathcal{K}, C) = (\mathcal{J}_1, \ldots, \mathcal{J}_m)$ be the function that returns all C-justifications w.r.t. $\mathcal{K}$. Such a set can be computed in exponential time in the size of C and $\mathcal{K}$ [18, 19].

In order to obtain a subset of $\mathcal{D}$ that is consistent with C, at least one statement from every justification has to be removed from $\mathcal{D}$. We focus on the strongest closure introduced in [8], which is minimal relevant closure.[3] By a preference of exceptionality rank, the removed statements shall be the rank-minimal[4] parts of all justifications. For any set of DCIs $\mathcal{U} \subseteq \mathcal{D}$, let $min_{r_{\mathcal{K}}}(\mathcal{U}) \subseteq \mathcal{U}$ contain those DCIs from $\mathcal{U}$ that are minimal w.r.t. $r_{\mathcal{K}}()$. For $\mathcal{K} = (\mathcal{T}, \mathcal{D})$, the subset of $\mathcal{D}$ that is consistent with a concept C for $justifications(\mathcal{K}, C) = (\mathcal{J}_1, \ldots, \mathcal{J}_m)$ for minimal relevant closure is $\mathcal{D}_C = \mathcal{D} \setminus (\bigcup_{i=1}^{m} min_{r_{\mathcal{K}}}(\mathcal{J}_i))$. $\mathcal{D}_X$ is therefore uniquely determined by the concept X for a given DKB.

Recall our notation from the Introduction to refer to combinations of different parameters in order to determine the semantics. We can use $\{rat, rel\} \times \{mat\}$ to characterise the materialisation-based entailment relations for subsumption. Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB and $cons_x()$ a function which determines for a DBox $\mathcal{D}$ and a concept C a consistent subset of the DBox to be used for reasoning, where the subset is picked according to the reasoning strength $x \in \{rat, rel\}$. We denote non-monotonic subsumption entailments obtained by materialisation-based reasoning as

$$\mathcal{K} \models^{(x,\mathrm{mat})} C \sqsubset\!\!\sim D \text{ iff } \overline{cons_x(\mathcal{D}, C)} \sqcap C \sqsubseteq_{\mathcal{T}} D.$$

For rational strength we define $cons_{\mathrm{rat}}(\mathcal{D}, C) = \mathcal{D}_i$ for $partition(\mathcal{D}) = (E_0, \ldots, E_n)$ where $0 \leq i \leq n$ is the least i s.t. $\mathcal{D}_i$ is consistent with C (as considered above). For relevant strength we consider $cons_{\mathrm{rel}}(\mathcal{D}, C) = \mathcal{D}_C$, as defined above.

Defeasible Instance Checking. Instance checking by materialisation is only considered under rational closure [4]. This algorithm requires the following about the DKB:

- $\mathcal{T}$ is *unfoldable*, i.e. only contains axioms of the form $A \sqsubseteq C$, $A = C$ for $A \in N_C$ such that each A occurs at most once on the left-hand side of an axiom and $\mathcal{T}$ is acyclic.
- $\mathcal{A}$ is *complete* which means in case of $\mathcal{EL}_\bot$:
  - $(C \sqcap D)(a) \in \mathcal{A} \implies C(a) \in \mathcal{A} \land D(a) \in \mathcal{A}$ and
  - $(\exists r.C)(a) \in \mathcal{A} \implies \exists b \in sig_I(\mathcal{A}).r(a, b) \in \mathcal{A} \land C(b) \in \mathcal{A}$.

The algorithm has a preprocessing phase which performs the well-separation of TBox and DBox and computes the rational DBox partition as described above. It unfolds the TBox, i.e. replace concept names A occurring in $\mathcal{A}$ and $\mathcal{D}$ for $A \sqsubseteq C \in \mathcal{T}$ by $C \sqcap A'$ (fresh $A' \in N_C$) and for $A{=}C \in \mathcal{T}$ by C. After this step the TBox can be omitted. Next, the ABox is transformed into a complete ABox. Their actual materialisation algorithm gets as input the complete ABox $\mathcal{A}$ and a DBox partition $partition(\mathcal{D})$.

The main idea is now to find the subset $\mathcal{D}_i$ of $\mathcal{D}$, for the least i s.t. $\mathcal{D}_i$ is still consistent with an individual $a \in sig_I(\mathcal{A})$. Consistency can be checked by adding the concept assertion of the material implication of $\mathcal{D}_i$ for a: $\mathcal{A}' = \mathcal{A} \cup \{\overline{\mathcal{D}_i}(a)\}$ and check for consistency of $\mathcal{A}'$. In case $\mathcal{D}_i$ is consistent with an individual a, the ABox is extended by the appropriate material implication concept assertion. However, since two individuals a and b might be explicitly connected in $\mathcal{A}$, e.g. $r(b, a) \in \mathcal{A}$, enriching individual b with defeasible information can influence the least i for which $\mathcal{D}_i$ is consistent with a after enriching b. For example, $\mathcal{A} = \{r(a, b), r(b, a)\}$, $\mathcal{T} = \{A \sqcap \exists r.B \sqsubseteq \bot\}$[5] and $\mathcal{D} = \{\top \sqsubset\!\!\sim A \sqcap B\}$. The DBox esentially states that everything is typically in concept A and B and the TBox enforces that no element in A is allowed to have an r-successors to an element in B. If a is enriched before b, then $A(a), B(a)$ is added to $\mathcal{A}$ and b cannot become an instance of B, since it is already an r-successor of a member of A. Dually, if b is enriched before a, then $A(b), B(b)$ is the only addition to the ABox.

[3] When convenient, we omit the word *minimal* in minimal relevant closure.

[4] Removing only the rank-minimal parts characterises minimal relevant closure, for a slightly different technique for the basic relevant closure see [8].

[5] The example is in $\mathcal{EL}_\bot$, however the TBox is not unfoldable. $\mathcal{T}' = \{A \sqsubseteq \forall r.(\neg B)\}$ however is unfoldable and semantically equivalent to $\mathcal{T}$, albeit in $\mathcal{ALC}$ syntax.

This motivates the addition of a sequence of the individuals in $sig_I(\mathcal{A})$ as a kind of preference relation to the knowledge base. This sequence determines in which order the individuals are enriched by defeasible information, where an element that appears "earlier" in the sequence can potentially be enriched with more defeasible information, as it is likely that less restrictions have been added to other individuals.

Let $s = (a_1, a_2, \ldots, a_n)$ be a (duplication-free) sequence of all elements in $sig_I(\mathcal{A})$. The reduction algorithm iterates over s and for each individual a_i it computes the least j such that $\mathcal{A} \cup \{\overline{\mathcal{D}_j}(a_i)\}$ is consistent and adds $\overline{\mathcal{D}_j}(a_i)$ to $\mathcal{A}$. The result is a so-called *default assumption extension* $\mathcal{A}^s_{rat}$ which is used to characterise a consequence relation

$$\mathcal{K}, s \models^{(rat,mat)} C(a) \;\; \textit{iff} \;\; \mathcal{A}^s_{rat} \models C(a).$$

3.1.1. Analysis and Discussion

The present approach to defeasible KLM-style reasoning in description logics is mainly motivated by the effect of neglecting quantified concepts when reasoning based on material implications. Moreover, we thoroughly investigate minimal relevant closure throughout the article, they are somewhat superior to rational closure w.r.t. the mentioned effect called *inheritance blocking*. We keep investigating rational strength at the same time, because resolving inheritance blocking through minimal relevant closure, comes at the cost of computational complexity, as it takes exponential time in the size of the knowledge base to compute all justifications of a given concept [18, 19]. Additionally, the relevant closures are known not to satisfy all KLM postulates [8], which can be interpreted as relevant entailment relations not being well behaved in KLM-terms. For practicality, rational closure may be considered strong enough while being computationally tractable in some sense.

The following example illustrates the problem of inheritance blocking occurring in rational closure, but not in minimal relevant closure, as well as the neglect of quantified concepts w.r.t. defeasible information.

Example 3.2. *Let* $\mathcal{K}_{ex1} = (\mathcal{T}_{ex1}, \mathcal{D}_{ex1})$ *with:*

$$\begin{aligned}
\mathcal{T}_{ex1} &= \{Boss \sqsubseteq Worker,\ Boss \sqcap \exists superior.Worker \sqsubseteq \bot\},\\
\mathcal{D}_{ex1} &= \{Worker \sqsubsim \exists superior.Boss,\ Worker \sqsubsim Productive,\ Boss \sqsubsim Responsible\},\ \textit{and}\\
partition(\mathcal{D}_{ex1}) &= \big(E_0 = \{Worker \sqsubsim \exists superior.Boss, Worker \sqsubsim Productive\},\\
&\qquad E_1 = \{Boss \sqsubsim Responsible\}\big).
\end{aligned}$$

When computing rational closure, the inconsistency $\overline{\mathcal{D}_{ex1}} \sqcap Boss \sqsubseteq_{\mathcal{T}_{ex1}} \bot$ *is detected, but* $\overline{\mathcal{D}_{ex1} \setminus E_0} \sqcap Boss \not\sqsubseteq_{\mathcal{T}_{ex1}} \bot$ *holds. Thus DKB* $\mathcal{K}_{ex1}$ *entails* $Boss \sqsubsim Worker \sqcap Responsible$*. DKB* $\mathcal{K}_{ex1}$ *does not entail* $Boss \sqsubsim Productive$*, even though the DCI* $Worker \sqsubsim Productive$ *does not cause the inconsistency of Boss.*

Under minimal relevant closure, $\mathcal{J}_1 = \{Worker \sqsubsim \exists superior.Boss\}$ *is the only Boss-justification w.r.t.* $\mathcal{K}_{ex1}$*. Therefore, the largest consistent DBox subset of* $\mathcal{D}_{ex1}$ *for reasoning about the concept Boss is the set*

$$\mathcal{D}' = \{Worker \sqsubsim Productive,\ Boss \sqsubsim Responsible\},$$

providing the consequence $\overline{\mathcal{D}'} \sqcap Boss \sqsubseteq_{\mathcal{T}_{ex1}} Productive$*.*

Example 3.2 also illustrates the short-coming of materialisation. Materialising the DCI $Worker \sqsubsim Productive$ to $\neg Worker \sqcup Productive$ in conjunction with $\exists superior.Worker$ yields a concept that is not subsumed by $\exists superior.Productive$. The defeasible information is unjustifiably disregarded when reasoning about quantified concepts yielding uniformly non-typical role successors. Hence, in Example 3.2, both rational and relevant closure (based on materialisation) are oblivious to the conclusion $Worker \sqsubsim \exists superior.Responsible$.

In case of instance checking, the materialisation-based approach can lead to similar problems regarding the quantified concepts. Since the ABox is unfolded w.r.t. the TBox, new existential restrictions can be introduced into the ABox. The transformation into a complete ABox ensures that the initially anonymous role successors are turned into named individuals for which then materialisation is performed. Thus the (newly) named individuals can, in principle, satisfy defeasible information. Now, since DCIs can introduce role-successors by existential restrictions, new anonymous individuals are generated for which materialisation is *not* performed. To illustrate this effect, consider $\mathcal{A} = \{A(a)\}$ and $\mathcal{D} = \{A \mathrel{\underset{\sim}{\sqsubset}} \exists r.C, C \mathrel{\underset{\sim}{\sqsubset}} D\}$. Since $\mathcal{A}$ is complete, the default assumption extension is

$$\mathcal{A}^s_{rat} = \mathcal{A} \cup \big\{ \big((\neg A \sqcup \exists r.C) \sqcap (\neg C \sqcup D)\big)(a) \big\} .$$

Clearly, $\mathcal{A}^s_{rat} \models \exists r.C(a)$, but $\mathcal{A}^s_{rat} \not\models \exists r.D(a)$ and thus $\mathcal{K} \not\models^{(rat,mat)} (\exists r.D)(a)$, even though nothing contradicts this. Hence, the criticism for ignoring quantified concepts persists regarding the materialisation-based approach.

Disregarding defeasible information for quantified concepts is essentially disregarding the full power of description logics in the realm of KLM-style defeasible reasoning. Therefore, our goal is devise algorithms that consider defeasible information fully for quantified concepts when deciding defeasible subsumption and instance checking. Furthermore, our approach requires less restrictions on the knowledge base for deciding defeasible instance checks.

3.2. Materialisation Adapted to $\mathcal{EL}_\bot$

Our approach to alleviate the discussed issues relies on the canonical model property of the DL $\mathcal{EL}_\bot$. We do not want to propose an entirely new (unrelated) semantics for defeasible reasoning in DL. Many semantics have been discussed in the literature and KLM-style preferential and rational reasoning is widely accepted. Therefore, we want the present approach to properly extend the results from [4, 8] on materialisation-based rational and relevant closure. This is achieved by first mimicking materialisation-based closures with our formalism and then strictly extending their semantic power in terms of coverage of objects with defeasible information. In order to combine the requirement of $\mathcal{EL}_\bot$ and the need to emulate materialisation-based reasoning. This section presents how to express materialisation-based reasoning in $\mathcal{EL}_\bot$ syntax, as concepts such as $\neg C \sqcup D$ are not in $\mathcal{EL}_\bot$ and we can obtain algorithms that are reduction algorithms using classical $\mathcal{EL}_\bot$ reasoning. Such reduction algorithms to classical reasoning are desirable in their own right, since they lend themselves to implementations. Casini et al. claim that materialisation-based reasoning resides in the same complexity as classical reasoning in the underlying DL, this however only (trivially) applies to Boolean DLs. The results in this article show that this is also the case for the sub-Boolean $\mathcal{EL}_\bot$.

We use the expressive power of GCIs and augment the TBox to express materialisation in $\mathcal{EL}_\bot$. The idea is to introduce a collection of new concepts for each original (sub-)concept C from the TBox, such that these auxiliary concepts need to satisfy defeasible information in the same fashion as the materialisation is enforcing it. Several of the results (on materialisation-based defeasible subsumption in $\mathcal{EL}_\bot$) will not be proven in detail here, as they are published in [11, 12] and elaborated on in full detail in [20].

3.2.1. Defeasible Subsumption by Materialisation Adapted to $\mathcal{EL}_\bot$

In the presence of the DBox $\{G \mathrel{\underset{\sim}{\sqsubset}} H\}$, materialisation determines subsumers of a concept F, by computing the subsumers of $(\neg G \sqcup H) \sqcap F$. To achieve the same effect, we introduce a fresh concept name $F_{\{G \mathrel{\underset{\sim}{\sqsubset}} H\}}$ that is intuitively a more typical subsumee of F. We require that

- R1 $F_{\{G \mathrel{\underset{\sim}{\sqsubset}} H\}}$ is a subclass of F, i.e. $F_{\{G \mathrel{\underset{\sim}{\sqsubset}} H\}} \sqsubseteq F$ is added to $\mathcal{T}$ and
- R2 $F_{\{G \mathrel{\underset{\sim}{\sqsubset}} H\}}$ satisfies $G \mathrel{\underset{\sim}{\sqsubset}} H$ in all models of the DKB, i.e. $F_{\{G \mathrel{\underset{\sim}{\sqsubset}} H\}} \sqcap G \sqsubseteq H$ is added to $\mathcal{T}$.

The extended TBox is satisfied, if all elements in $F_{\{G \mathrel{\underset{\sim}{\sqsubset}} H\}}$ either do not belong to G or do belong to H. We define an extension of the TBox, in order to be able to characterise the desired (more typical) subclass of F w.r.t. a whole set of DCIs $\mathcal{D}$. Let $N_C^{aux} \subseteq N_C$ be a set of concept names that do not occur in DKB $\mathcal{K}$.

Definition 3.3 (extended TBox). Let F be a concept, $F_\mathcal{D} \in N_C^{aux}$, $\mathcal{T}$ be a TBox $\mathcal{T}$, and $\mathcal{D}$ be a DBox $\mathcal{D}$. The *extended TBox* of F w.r.t. $\mathcal{D}$ is:

$$\mathcal{T}_\mathcal{D}(F) \ = \ \mathcal{T} \cup \{F_\mathcal{D} \sqsubseteq F\} \cup \{F_\mathcal{D} \sqcap G \sqsubseteq H \mid G \mathrel{\underset{\sim}{\sqsubset}} H \in \mathcal{D}\}. \tag{1}$$

In this definition $\{F_\mathcal{D} \sqsubseteq F\}$ ensures that requirement R1 is fulfilled. The last set of GCIs in Eq. (1) ensures that every instance of $F_\mathcal{D}$ satisfies requirement R2 for *all* the DCIs in $\mathcal{D}$. Clearly, the extended TBox $\mathcal{T}_\mathcal{D}(F)$ is an $\mathcal{EL}_\bot$ TBox if $\mathcal{T}$ is one. We want to show that classical reasoning w.r.t. $\mathcal{T}_\mathcal{D}(F)$ produces equivalent results to reasoning with defeasible information using the materialisation-based approach.

As a first intermediate result, it holds that the auxiliary concept $F_\emptyset$ from the extended TBox $\mathcal{T}_\emptyset(F)$ and the concept F from $\mathcal{T}$ have the same subsumers w.r.t. the signature of $\mathcal{T}$ and $\mathcal{D}$. The intuition is that without any material implications to the extended TBox, reasoning about the auxiliary concept $F_\emptyset$ coincides with reasoning about the original concept F. This property supports the proof of the main result for equivalence of materialisation-based subsumption in $\mathcal{ALC}$ and $\mathcal{EL}_\bot$, where it acts as the induction start for the induction on the size of the set of DCIs $\mathcal{D}$.

Proposition 3.4. *Let $\mathcal{T}$ be a TBox and F, G be concepts with $sig(G) \cap N_C^{aux} = \emptyset$. Then $F \sqsubseteq_\mathcal{T} G$ iff $F_\emptyset \sqsubseteq_{\mathcal{T}_\emptyset(F)} G$.* □ *in [20]*

It only remains to show that subsumees of a concept $F_\mathcal{D}$ (by classical semantics) w.r.t. the TBox $\mathcal{T}_\mathcal{D}(F)$ coincide with the classical subsumees of $\overline{\mathcal{D}} \sqcap F$ for arbitrary sets of DCIs $\mathcal{D}$. Since the definition of canonical models relies on classical reasoning in $\mathcal{EL}_\bot$, this result provides the means to define a canonical model using (several) TBoxes $\mathcal{T}_\mathcal{D}(F)$ and obtain subsumption consequences that are equivalent to those obtainable through materialisation (c.f. Section 4).

Lemma 3.5. *Let $\mathcal{T}$ be a TBox $\mathcal{T}$, $\mathcal{D}$ a DBox, and C, D be concepts, with $sig(X) \cap N_C^{aux} = \emptyset$ for $X \in \{\mathcal{T}, \mathcal{D}, C, D\}$. Then*

$$\overline{\mathcal{D}} \sqcap C \sqsubseteq_\mathcal{T} D \text{ iff } C_\mathcal{D} \sqsubseteq_{\mathcal{T}_\mathcal{D}(C)} D.$$

Proof (sketch). The lemma is proven by induction on the size of $\mathcal{D}$. The base case is $\mathcal{D} = \emptyset$ and thereby Prop. 3.4 holds. For the induction step, let $\mathcal{D}' = \mathcal{D} \cup \{G \mathrel{\underset{\sim}{\sqsubset}} H\}$ and use the hypothesis that the claim holds for $\mathcal{D}$. We do a case distinction for (i) $\overline{\mathcal{D}} \sqcap C \sqsubseteq G$ and (ii) $\overline{\mathcal{D}} \sqcap C \not\sqsubseteq G$. For case (i), it can be shown that reasoning with $C \sqcap H$, yields the same consequences as reasoning with C. All elements in $C \sqcap H$ satisfying G, already satisfy H. Thus we can remove the introduced DCI $G \mathrel{\underset{\sim}{\sqsubset}} H$ without losing any consequences. Hence, $G \mathrel{\underset{\sim}{\sqsubset}} H$ can be removed from $\mathcal{D}'$ and the induction hypothesis holds. In case (ii) we show that the added DCI has no effect on the reasoning. By the condition of this case, no element in C, satisfying all DCIs in D, satisfies G. Therefore, the presence of the DCI $G \mathrel{\underset{\sim}{\sqsubset}} H$ does not affect the reasoning allowing it to be removed in order to reduce the induction step to the induction hypothesis. In both cases, both sides of the "*iff*" in the claim are reduced to their respective sides of the induction hypothesis individually. The full proof is fairly long and technical and the interested reader is referred to [20]. □

Lemma 3.5 directly implies that after determining $\mathcal{D}' = cons_x(\mathcal{D}, C)$ (for querying $C \mathrel{\underset{\sim}{\sqsubset}} D$ under the strength x), both, the materialisation and the TBox extension approach are capable to produce the same reasoning results w.r.t. $\mathcal{D}'$. This naturally shows that materialisation-based rational reasoning remains polynomial for $\mathcal{EL}_\bot$ DKBs, which is not a trivial consequence of the approach in [4]. Note that, for computing rational closure, the consistency of $\mathcal{D}_i$ is checked in [4] using materialisation as well. In order to remain entirely in $\mathcal{EL}_\bot$, the consistency check $\overline{\mathcal{D}_i} \sqcap C \sqsubseteq_\mathcal{T} \bot$ needs to be reformulated to $C_{\mathcal{D}_i} \sqsubseteq_{\mathcal{T}_{\mathcal{D}_i}(C)} \bot$ which clearly yields the same result by Lemma 3.5. For relevant closure, $\mathcal{D}_C$ is simply determined by appropriate justification algorithms as a preprocessing step to both approaches, hence allowing the same results under relevant closure, though the computational complexity is dominated by the computation of all justifications [18, 19].

3.2.2. Defeasible Instance Checking by Materialisation Adapted to $\mathcal{EL}_\bot$

Here we develop an algorithm that yields the same results as the algorithm from [4], but using $\mathcal{EL}_\bot$ only. Our algorithm proceeds by extending the TBox by auxiliary concepts which encode the DCIs from the input DBox $\mathcal{D}$. These concepts are used to assert in $\mathcal{A}$ that individuals satisfy the corresponding DCI. Introduce fresh concept names $D_{E \mathrel{\underset{\sim}{\sqsubset}} F} \in N_C^{aux}$ and let $\mathcal{T}_\mathcal{D} = \{D_{E \mathrel{\underset{\sim}{\sqsubset}} F} \sqcap E \sqsubseteq F \mid E \mathrel{\underset{\sim}{\sqsubset}} F \in \mathcal{D}\}$. Clearly, $\mathcal{T} \cup \mathcal{T}_\mathcal{D}$ is a conservative extension of $\mathcal{T}$, thus if $sig(C, D) \subseteq sig(\mathcal{T})$ then $C \sqsubseteq_{\mathcal{T}} D$ *iff* $C \sqsubseteq_{\mathcal{T}\cup\mathcal{T}_{\mathcal{D}}} D$ follows. While the algorithm from [4] extends the ABox by assertions of the form $(\neg E \sqcup F)(a)$, our algorithm extends $\mathcal{A}$ by assertions of the form $D_{E \precsim F}(a)$, ensuring that the GCI $D_{E \precsim F} \sqcap E \sqsubseteq F$ "can trigger" for the individual a. It is important to note that our algorithm for $\mathcal{EL}_\bot$ does not require the restrictions on $\mathcal{A}$ and $\mathcal{T}$ that are imposed in [4] and is thus more general.

Next, we show that a knowledge base that contains a materialisation-based extension of the ABox can be translated into an equivalent $\mathcal{EL}_\bot$ KB using the GCIs and the concept assertions just mentioned. To address these two kinds of extensions of ABoxes, we introduce some notation. For a given set of DCIs $\mathcal{D}$, an individual a and fresh concept names $D_{E \precsim F}$, define

- $\mathcal{A}^a_{\mathcal{D}} = \{(\neg E \sqcup F)(a) \mid E \precsim F \in \mathcal{D}\}$, and
- $\widehat{\mathcal{A}}^a_{\mathcal{D}} = \{D_{E \precsim F}(a) \mid E \precsim F \in \mathcal{D}\}$.

Lemma 3.6. *Let $K = (\mathcal{A}, \mathcal{T})$ be a classical $\mathcal{EL}_\bot$ KB, $\mathcal{D}$ be a DBox, C a concept with $sig(C) \subseteq sig(\mathcal{A}, \mathcal{T})$, $b \in sig_I(\mathcal{A})$ an individual, and $\Pi : sig_I(\mathcal{A}) \to 2^{\mathcal{D}}$ a function. Then the following two statements are equivalent*

1. $(\mathcal{A} \cup \bigcup_{a\in sig_I(\mathcal{A})} \mathcal{A}^a_{\Pi(a)}, \mathcal{T}) \models C(b)$
2. $(\mathcal{A} \cup \bigcup_{a\in sig_I(\mathcal{A})} \widehat{\mathcal{A}}^a_{\Pi(a)}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}) \models C(b)$.

Proof. The size of the function Π is defined as $|\Pi| = \sum_{a\in sig_I(\mathcal{A})} |\Pi(a)|$. We prove the claim by induction on $|\Pi|$. The base case for $|\Pi| = 0$ assigns $\Pi(a) = \emptyset$ for all $a \in sig_I(\mathcal{A})$, it follows that $\mathcal{A}^a_{\Pi(a)}$ and $\widehat{\mathcal{A}}^a_{\Pi(a)}$ are empty. Since $\mathcal{T}\cup\mathcal{T}_{\mathcal{D}}$ is a conservative extension of $\mathcal{T}$, the claim holds for $sig(C) \subseteq sig(\mathcal{A}, \mathcal{T})$. For the induction hypothesis, let $|\Pi| = n$ and assume the claim holds for Π. In the induction step there is some $a \in sig_I(\mathcal{A})$, s.t. $\Pi'(a) = \Pi(a)\cup\{E \precsim F\}$ (where $E \precsim F \notin \Pi(a)$) and for all $b \in sig_I(\mathcal{A})$ $b \neq a$, $\Pi'(b) = \Pi(b)$. Clearly $|\Pi'| = |\Pi| + 1$ and $\mathcal{A}^a_{\Pi'(a)} = \mathcal{A}^a_{\Pi(a)} \cup \{(\neg E \sqcup F)(a)\}$ as well as $\mathcal{A}^b_{\Pi'(b)} = \mathcal{A}^b_{\Pi(b)}$ for $b \neq a$ (analogous for $\widehat{\mathcal{A}}^a_{\Pi'(a)}$). We show in the induction step (IS) that the following statements are equivalent for arbitrary b

1. $(\mathcal{A} \cup \bigcup_{b\in sig_I(\mathcal{A})} \mathcal{A}^b_{\Pi(b)} \cup \{(\neg E \sqcup F)(a)\}, \mathcal{T}) \models C(b)$
2. $(\mathcal{A} \cup \bigcup_{b\in sig_I(\mathcal{A})} \widehat{\mathcal{A}}^b_{\Pi(b)} \cup \{D_{E \precsim F}(a)\}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}) \models C(b)$

We distinguish two cases, depending on whether $E(a)$ is entailed or not.

Case 1: $(\mathcal{A}\cup\bigcup_{b\in sig_I(\mathcal{A})} \mathcal{A}^b_{\Pi(b)}, \mathcal{T}) \models E(a)$**.** By IH, this is equivalent to $(\mathcal{A}\cup\bigcup_{b\in sig_I(\mathcal{A})} \widehat{\mathcal{A}}^b_{\Pi(b)}, \mathcal{T}\cup\mathcal{T}_{\mathcal{D}}) \models E(a)$. Monotonicity of classical reasoning implies $(\mathcal{A} \cup \bigcup_{b\in sig_I(\mathcal{A})} \mathcal{A}^b_{\Pi(b)} \cup \{(\neg E \sqcup F)(a)\}, \mathcal{T}) \models E(a)$ and therefore the knowledge base $(\mathcal{A} \cup \bigcup_{b\in sig_I(\mathcal{A})} \mathcal{A}^b_{\Pi(b)} \cup \{(\neg E \sqcup F)(a)\}, \mathcal{T})$ has exactly the same models as

$$(\mathcal{A} \cup \bigcup_{b\in sig_I(\mathcal{A})} \mathcal{A}^b_{\Pi(b)} \cup \{F(a)\}, \mathcal{T}).$$

Since no restrictions (other than using $\mathcal{EL}_\bot$ syntax) were imposed on the ABox $\mathcal{A}$, the induction hypothesis holds also when using the ABox $\mathcal{A}\cup\{F(a)\}$.

Towards reducing Statement 2 of the induction step (IS) to the induction hypothesis, monotonicity and the case assumption imply $(\mathcal{A}\cup\bigcup_{b\in sig_I(\mathcal{A})} \widehat{\mathcal{A}}^b_{\Pi(b)}\cup\{D_{E \precsim F}(a)\}, \mathcal{T}\cup\mathcal{T}_{\mathcal{D}}) \models E(a)$. Since $D_{E \precsim F}\sqcap E \sqsubseteq F \in \mathcal{T}_{\mathcal{D}}$, it follows that the same knowledge base also entails $F(a)$. Therefore, $(\mathcal{A}\cup\bigcup_{b\in sig_I(\mathcal{A})} \widehat{\mathcal{A}}^b_{\Pi(b)}\cup\{D_{E \precsim F}(a)\}, \mathcal{T}\cup\mathcal{T}_{\mathcal{D}})$ has exactly the same models as $(\mathcal{A} \cup \bigcup_{b\in sig_I(\mathcal{A})} \widehat{\mathcal{A}}^b_{\Pi(b)} \cup \{F(a)\}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}})$ which reduces Statement 2 of (IS) to the IH using the ABox $\mathcal{A}\cup\{F(a)\}$ which shows equivalence of 1 and 2 (IS) in Case 1.

Case 2: $(\mathcal{A} \cup \bigcup_{b\in sig_I(\mathcal{A})} \mathcal{A}^b_{\Pi(b)}, \mathcal{T}) \not\models E(a)$**.** This is equivalent to $(\mathcal{A} \cup \bigcup_{b\in sig_I(\mathcal{A})} \widehat{\mathcal{A}}^b_{\Pi(b)}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}) \not\models E(a)$ by IH. We show that the extension in Π' has no influence on the entailed consequences. The added assertion can therefore be removed without changing the set of models, allowing to directly apply the induction hypothesis. Towards the reduction of 1 in (IS) to the hypothesis, we show that

$$(\mathcal{A} \cup \bigcup_{b \in sig_I(\mathcal{A})} \mathcal{A}^b_{\Pi(b)} \cup \{(\neg E \sqcup F)(a)\}, \mathcal{T}) \models C(b) \textit{ iff } (\mathcal{A} \cup \bigcup_{b \in sig_I(\mathcal{A})} \mathcal{A}^b_{\Pi(b)}, \mathcal{T}) \models C(b).$$

The *if* direction follows trivially from monotonicity of classical reasoning. For the *only-if* direction assume for a contradiction that $(\mathcal{A} \cup \bigcup_{b \in sig_I(\mathcal{A})} \mathcal{A}^b_{\Pi(b)}, \mathcal{T}) \not\models C(b)$. Then, there must be a model $\mathcal{I}$ of $(\mathcal{A} \cup \bigcup_{b \in sig_I(\mathcal{A})} \mathcal{A}^b_{\Pi(b)}, \mathcal{T})$ s.t. $b^{\mathcal{I}} \notin C^{\mathcal{I}}$ and by the case assumption a model $\mathcal{J}$ of $(\mathcal{A} \cup \bigcup_{b \in sig_I(\mathcal{A})} \mathcal{A}^b_{\Pi(b)}, \mathcal{T})$ s.t. $a^{\mathcal{J}} \notin E^{\mathcal{J}}$. Since models of $\mathcal{EL}_\bot$ knowledge bases are closed under product, $\mathcal{I} \times \mathcal{J}$ is a model of the same KB s.t. neither $C(b)$ nor $E(a)$ are satisfied in $\mathcal{I} \times \mathcal{J}$. $\mathcal{I} \times \mathcal{J}$ clearly satisfies $(\neg E \sqcup F)(a)$ and is therefore a model of $(\mathcal{A} \cup \bigcup_{b \in sig_I(\mathcal{A})} \mathcal{A}^b_{\Pi(b)} \cup \{(\neg E \sqcup F)(a)\}, \mathcal{T})$ contradicting our assumption $(\mathcal{A} \cup \bigcup_{b \in sig_I(\mathcal{A})} \mathcal{A}^b_{\Pi(b)} \cup \{(\neg E \sqcup F)(a)\}, \mathcal{T}) \models C(b)$.

Towards the reduction of 2 in (IS) to the induction hypothesis, we show that

$$(\mathcal{A} \cup \bigcup_{b \in sig_I(\mathcal{A})} \widehat{\mathcal{A}}^b_{\Pi(b)} \cup \{D_{E \mathrel{\underset{\sim}{\sqsubset}} F}(a)\}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}) \models C(b) \textit{ iff } (\mathcal{A} \cup \bigcup_{b \in sig_I(\mathcal{A})} \widehat{\mathcal{A}}^b_{\Pi(b)}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}) \models C(b).$$

The *if* direction is again trivial by monotonicity and we prove the *only-if* direction by a contradiction. Assume $(\mathcal{A} \cup \bigcup_{b \in sig_I(\mathcal{A})} \widehat{\mathcal{A}}^b_{\Pi(b)}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}) \not\models C(b)$. Again, by the closure of models for $\mathcal{EL}_\bot$ knowledge bases under cross-product, there exists a model $\mathcal{I}$ of $(\mathcal{A} \cup \bigcup_{b \in sig_I(\mathcal{A})} \widehat{\mathcal{A}}^b_{\Pi(b)}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}})$ s.t. $\mathcal{I} \not\models E(a)$ and $\mathcal{I} \not\models C(b)$. Let $\mathcal{J} = \mathcal{I}[D_{E \mathrel{\underset{\sim}{\sqsubset}} F} / D^{\mathcal{I}}_{E \mathrel{\underset{\sim}{\sqsubset}} F} \cup \{a^{\mathcal{I}}\}]$. Since $D_{E \mathrel{\underset{\sim}{\sqsubset}} F}$ *only* appears in $\mathcal{T} \cup \mathcal{T}_{\mathcal{D}}$ in the GCI $D_{E \mathrel{\underset{\sim}{\sqsubset}} F} \sqcap E \sqsubseteq F$, it is not hard to see that

- $\mathcal{J} \models \mathcal{A} \cup \bigcup_{b \in sig_I(\mathcal{A})} \widehat{\mathcal{A}}^b_{\Pi(b)}$,
- $\mathcal{J} \models D_{E \mathrel{\underset{\sim}{\sqsubset}} F}(a)$, and
- $\mathcal{J} \models (\mathcal{T} \cup \mathcal{T}_{\mathcal{D}}) \setminus \{D_{E \mathrel{\underset{\sim}{\sqsubset}} F} \sqcap E \sqsubseteq F\}$

hold. It also follows from the construction of $\mathcal{J}$, that $X^{\mathcal{J}} = X^{\mathcal{I}}$ for $\mathcal{EL}_\bot$ concepts X with $D_{E \mathrel{\underset{\sim}{\sqsubset}} F} \notin sig(X)$. Therefore, $a^{\mathcal{J}} \notin E^{\mathcal{J}}$ implies $(D_{E \mathrel{\underset{\sim}{\sqsubset}} F} \sqcap E)^{\mathcal{J}} = (D_{E \mathrel{\underset{\sim}{\sqsubset}} F} \sqcap E)^{\mathcal{I}}$ (as well as $F^{\mathcal{J}} = F^{\mathcal{I}}$), showing

$$\mathcal{J} \models (\mathcal{A} \cup \bigcup_{b \in sig_I(\mathcal{A})} \widehat{\mathcal{A}}^b_{\Pi(b)} \cup \{D_{E \mathrel{\underset{\sim}{\sqsubset}} F}(a)\}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}).$$

Now, $\mathcal{J} \not\models C(b)$ contradicts the premise of the only-if direction. From the result that the extension of the ABox has no effect on the set of entailments (in both sides), it follows that reasoning is equivalent to using the induction hypothesis, which proves the induction step to be true under Case 2. □

By Lemma 3.6, we know, given the analogous materialisation-based extensions of the ABox (for $\mathcal{ALC}$ and $\mathcal{EL}_\bot$), that classical instance checking, if applied to $\mathcal{EL}_\bot$ KBs, using both ABox extensions respectively, provides the same results. It remains to introduce an algorithm that computes the default assumption extension ABox from an $\mathcal{EL}_\bot$ DKB $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ by means of classical reasoning in $\mathcal{EL}_\bot$. This algorithm does not require the restrictions on the TBox or the ABox as in the algorithm from [4]. We only assume w.l.o.g. that no conjunction appears on the top-level of concepts in the ABox to simplify notation. We denote by $\widehat{\mathcal{A}}^s_{rat}$ the ABox obtained from Algorithm 1 for the input $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ and the sequence s on $sig_I(\mathcal{A})$.

The following lemma shows for materialisation-based rational instance checking for $\mathcal{EL}_\bot$ KBs (under the restrictions imposed on $\mathcal{A}$ and $\mathcal{T}$ in [4]) equivalence between using $\mathcal{ALC}$ for syntax and reasoning as in [4] and using $\mathcal{EL}_\bot$ for syntax and reasoning as introduced above. One can easily produce an example using a general TBox and a (conjunction-free) ABox which cannot be processed by the procedure introduced in [4].

Algorithm 1: Computation of default assumption extension $\widehat{\mathcal{A}}^s_{rat}$

Input: $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ ($\mathcal{A}$ is conjunction-free), sequence $(a_1, \ldots, a_n)$ of all individuals in $\mathcal{A}$,
$partition(\mathcal{D}) = (E_0, \ldots, E_m)$

Output: Default assumption extension $\widehat{\mathcal{A}}^s_{rat}$

1 **for** a_i *in* $(a_1, \ldots, a_n)$ **do**
2   **for** E_j *in* $(E_0, \ldots, E_m)$ **do**
3         $\mathcal{D}_j := \bigcup_{k=j}^{m} E_k$
4   **if** $\big(\mathcal{A} \cup \{\mathcal{D}_{E \mathrel{\underset{\sim}{\sqsubset}} F}(a_i) \mid E \mathrel{\underset{\sim}{\sqsubset}} F \in \mathcal{D}_j\},\ \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}\big)$ *is consistent* **then**
5             $\mathcal{A} := \mathcal{A} \cup \{\mathcal{D}_{E \mathrel{\underset{\sim}{\sqsubset}} F}(a_i) \mid E \mathrel{\underset{\sim}{\sqsubset}} F \in \mathcal{D}_j\}$
6   **exit loop**
7   **end**
8   **end**
9 **end**
10 **return** $\mathcal{A}$

Lemma 3.7. *Let $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ be an $\mathcal{EL}_\bot$ DKB, with a complete ABox $\mathcal{A}$ and an unfoldable TBox $\mathcal{T}$. Let $\mathcal{A}^s_{rat}$ and $\widehat{\mathcal{A}}^s_{rat}$ be default assumption extensions, and C be an $\mathcal{EL}_\bot$ concept with $sig(C) \subseteq sig(\mathcal{K})$. Then the following holds*

$$\mathcal{K}, s \models^{(rat,mat)} C(a) \iff (\widehat{A}_s, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}) \models C(a).$$

Proof. By definition, $\mathcal{K}, s \models^{(rat,mat)} C(a)$ holds *iff* $\mathcal{A}^s_{rat} \models C(a)$ by classical semantics. Let $\mathcal{A}^s_{rat} = \mathcal{A} \uplus \mathcal{A}'$ be the ABox extension as obtained from the algorithm described in [4], where $\mathcal{A}'$ contains only assertions of the form $(\neg E \sqcup F)(a)$ for $E \mathrel{\underset{\sim}{\sqsubset}} F \in \mathcal{D}$ and $a \in sig_I(\mathcal{A})$. If $\mathcal{T}$ is an unfoldable TBox, $\mathcal{A}$ a complete ABox unfolded w.r.t. $\mathcal{T}$ and $\mathcal{D}$ a DBox unfolded w.r.t. $\mathcal{T}$, then $(\mathcal{A}^s_{rat}, \emptyset) \models C(a)$ *iff* $(\mathcal{A}^s_{rat}, \mathcal{T}) \models C(a)$ holds and is not hard to check. It is obvious that the two for-loops in Algorithm 1 in Lines 1 and 2 are the same as in the Algorithm in [4]. Within those for-loops, a set of potential assertions is considered for extending the ABox in both algorithms. Let this set be $\mathcal{A}_1$ in Algorithm 1 and $\mathcal{A}_2$ in the Algorithm in [4]. Since both algorithms use the same sequence $(a_0, \ldots, a_n)$ and the same $partition(\mathcal{D})$ and thus the same sequence $(E_0, \ldots, E_m)$, it holds that $D_{E \mathrel{\underset{\sim}{\sqsubset}} F}(a) \in \mathcal{A}_1$ *iff* $(\neg E \sqcup F)(a) \in \mathcal{A}_2$. Both algorithms proceed with a consistency check for the extension of the *current* ABox with the potential extension $\mathcal{A}_1$, $\mathcal{A}_2$ (resp.). This consistency check has the same outcome in every iteration (for both for-loops), due to Lemma 3.6 and the inductive nature of the extension algorithms. It follows that for the final ABox extensions $\mathcal{A}^s_{rat} = \mathcal{A} \uplus \mathcal{A}'$ and $\widehat{\mathcal{A}}^s_{rat} = \mathcal{A} \uplus \mathcal{A}''$, that $(\neg E \sqcup F)(a) \in \mathcal{A}'$ *iff* $D_{E \mathrel{\underset{\sim}{\sqsubset}} F}(a) \in \mathcal{A}''$. Then we can apply Lemma 3.6 and it follows that $(\mathcal{A}^s_{rat}, \mathcal{T}) \models C(a)$ *iff* $(\widehat{\mathcal{A}}^s_{rat}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}) \models C(a)$. □

With Lemma 3.7 we have shown that the Algorithms for deciding instance checking in defeasible $\mathcal{EL}_\bot$ (w.r.t. the imposed restrictions) under rational closure from [4] and the one developed in this subsection yield the same answers. Unfortunately the combination of unfoldable TBoxes and $\mathcal{EL}_\bot$ syntax is very restrictive, i.e. disjointness $C \sqcap D \sqsubseteq \bot$ cannot be expressed in $\mathcal{T}$, hence the consistency checks in the algorithms and thus the construction of $\mathcal{A}^s_{rat}$ and $\widehat{\mathcal{A}}^s_{rat}$ immediately become trivial.

In this section we have developed algorithms for deciding subsumption under rational and minimal relevant closure and instance checking under rational closure by a reduction to classical $\mathcal{EL}_\bot$ reasoning. For rational closure, both reductions are polynomial in the size of the DKB, whereas for subsumption under minimal relevant closure, the computational complexity is dominated by the computation of justifications. Instance checking under relevant closure was never discussed prior to this article (c.f. Section 5). This confirms the claim from Casini et al. in [4] that defeasible reasoning does not add computational complexity even for sub-Boolean DLs: our reduction shows that reasoning for defeasible $\mathcal{EL}_\bot$ remains polynomial (for rational closure). Besides this pleasing result, this algorithm still bears the deficits regarding defeasible information for the (nested) role successors diagnosed in Section 3.1.1 for the materialisation-based approach. This deficit is what we remedy by reasoning w.r.t. a new kind of interpretation.

4. Typicality Interpretations for Deciding Defeasible Subsumption

In this section we investigate defeasible subsumption in several settings. Again we denote the considered semantics by a pair over strength and coverage, i.e., a pair from

$$\{\text{rat}, \text{rel}\} \times \{\text{prop}, \text{nest}\},$$

where, for instance, (rat, prop) refers to rational semantics of propositional nature and (rel, nest) refers to relevant semantics including defeasible entailments for nested existential restrictions. We consider entailments obtained by materialisation to be of *propositional nature w.r.t. defeasible information.*[6] Now, since the use of roles (to characterise concepts) is the main asset of DLs, a KLM-style non-monotonic system for DLs, should allow for defeasible conclusions in nested concepts. There are now 4 combinations from {rat, rel} × {prop, nest} to be investigated for subsumption which we address in the following order. First we investigate the simple inferences of propositional coverage and then extend it to nested coverage. Since the techniques and theoretical results required for relevant semantics are a generalisation of those needed for rational semantics, we start by investigating the general case of relevant reasoning.

Canonical models for classical $\mathcal{EL}$ have domain elements as representatives for concepts and individuals from the KB. To extend this kind of model to defeasible variants of $\mathcal{EL}$, the idea is to use concept or individual representatives that satisfy different sets of defeasible concept inclusions and thus represent different "amounts of typicality". Our goal is to construct the models such that the defeasible subsumption relationships that follow under the chosen semantics can be directly read-off from the model. We define such models and show some of their properties in preparation of the investigations for the four varieties of defeasible subsumption.

4.1. Introducing Typicality Interpretations

The kind of interpretations we introduce has domains essentially structured by a 2-dimensional grid, where one dimension are the representatives for concepts and individuals and the second dimension captures sets of DCIs from the DBox and thus represent different amounts of typicality. Depending on the strength of reasoning considered, different subsets of the DBox are used in such a typicality interpretation. In typicality interpretations only those points in the grid are instantiated by elements, where the combination of the concept and the DBox subset are a consistent combination. Observe that this kind of models also captures the classical case, i.e., if the DBox is empty, then the second dimension has only one entry: the empty set. This "level of no typicality" is included in every typicality model.

We define typicality interpretations formally. In order to cater for different strengths of reasoning in the semantics, we distinguish different structures of how the subsets from $\mathcal{D}$ relate to each other; this determines the "shape of the interpretation domain".

Definition 4.1. Let $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ be a DKB. The domain $\Delta^{\mathcal{K}}$ is a *typicality domain over* $\mathcal{K}$ if all domain elements are of the form d_a or $d_F^{\mathcal{U}}$, where

- $a \in sig_I(\mathcal{A})$,
- $F \in Qc(\mathcal{K})$,
- $\mathcal{U} \subseteq \mathcal{D}$, and
- $\{d_F^{\emptyset} \mid F \in Qc(\mathcal{K})\} \subseteq \Delta^{\mathcal{K}}$.

[6] This observation is somewhat supported by the translation of propositional KLM postulates to DL syntax without considering quantified concepts in [4].

The *set of represented subsets of* $\mathcal{D}$ in $\Delta^{\mathcal{K}}$ is $\Gamma(\Delta^{\mathcal{K}}) = \{\mathcal{U} \subseteq \mathcal{D} \mid \exists d_F^{\mathcal{U}} \in \Delta^{\mathcal{K}}\}$. The *shape of a typicality domain* is that of a

- *chain*, if $\Gamma(\Delta^{\mathcal{K}})$ is totally ordered by $\subseteq$,
- *lattice*, if no further restrictions are imposed on $\Gamma(\Delta^{\mathcal{K}})$.

A typicality domain consists of only two types of domain elements: *concept representatives* are associated with an $\mathcal{EL}_\bot$ concept and a subset of the DBox (e.g. $d_F^{\mathcal{U}}$) and *individual representatives* are associated with an individual from the ABox.

Definition 4.2. An interpretation $\mathcal{I} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{K}}$ is a typicality domain over some DKB $\mathcal{K}$, is a *typicality interpretation*.

We single out those typicality interpretations whose representatives of a concept really are in the extension of that concept and elements that belong to existential restrictions really have the required role-successor (in the required concept) and on the level of no typicality. Thus such a role-successor satisfies at least the TBox.

Definition 4.3. A typicality interpretation $\mathcal{I} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{I}})$ over a DKB $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ is called *standard iff*

1. $d_F^{\mathcal{U}} \in F^{\mathcal{I}}$ for $F \in Qc(\mathcal{K})$ and $\mathcal{U} \subseteq \mathcal{D}$,
2. $d \in (\exists r.E)^{\mathcal{I}} \implies (d, d_E^{\emptyset}) \in r^{\mathcal{I}}$ for $E \in Qc(\mathcal{K})$, $d \in \Delta^{\mathcal{K}}$

Notice that Condition 2 considers any domain element d, hence d could either be a concept representative or an individual representative. A crucial consequence for standard typicality interpretations is that they are well-behaved regarding intersections of interpretations (Definition 2.2).

Proposition 4.4. *Let $\mathcal{I} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{I}})$ and $\mathcal{J} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{J}})$ be typicality interpretations over the DKB $\mathcal{K}$.*

1. $\mathcal{I} \subseteq \mathcal{J}$ implies $C^{\mathcal{I}} \subseteq C^{\mathcal{J}}$ for all $\mathcal{EL}_\bot$ concepts C,.

2. If $\mathcal{I}$ and $\mathcal{J}$ are standard, then $C^{\mathcal{I}\cap\mathcal{J}} = C^{\mathcal{I}} \cap C^{\mathcal{J}}$ for all $\mathcal{EL}_\bot$ concepts C with $Qc(C) \subseteq Qc(\mathcal{K})$.

Proof. We prove Claim 1 by induction on the structure of C. The induction start $C = A$ ($A \in N_C$) follows by definition of $\subseteq$ for interpretations over a shared domain and is trivial for $C = \bot$. Assume $E^{\mathcal{I}} \subseteq E^{\mathcal{J}}$ and $F^{\mathcal{I}} \subseteq F^{\mathcal{J}}$. It is clear that $(E \sqcap F)^{\mathcal{I}} \subseteq (E \sqcap F)^{\mathcal{J}}$. For $C = \exists r.E$, the premise $\mathcal{I} \subseteq \mathcal{J}$ implies $r^{\mathcal{I}} \subseteq r^{\mathcal{J}}$ and together with the induction hypothesis we obtain

$$\begin{aligned}(\exists r.E)^{\mathcal{I}} &= \{d \in \Delta^{\mathcal{K}} \mid \exists (d,e) \in r^{\mathcal{I}}.e \in E^{\mathcal{I}}\} \\ &\subseteq \{d \in \Delta^{\mathcal{K}} \mid \exists (d,e) \in r^{\mathcal{J}}.e \in E^{\mathcal{J}}\} \\ &= (\exists r.E)^{\mathcal{J}}.\end{aligned}$$

We show Claim 2 also by induction on the structure of C. The case of $C = A$ ($A \in N_C$) follows from the definition of $\mathcal{I}\cap\mathcal{J}$, i.e. $A^{\mathcal{I}\cap\mathcal{J}} = A^{\mathcal{I}} \cap A^{\mathcal{J}}$ and for $C = \bot$ the claim trivially holds. Assume $X^{\mathcal{I}\cap\mathcal{J}} = X^{\mathcal{I}} \cap X^{\mathcal{J}}$ ($X \in \{E, F\}$) and let $C = E \sqcap F$. Then

$$\begin{aligned}(E \sqcap F)^{\mathcal{I}\cap\mathcal{J}} &= E^{\mathcal{I}\cap\mathcal{J}} \cap F^{\mathcal{I}\cap\mathcal{J}} \\ &= E^{\mathcal{I}} \cap E^{\mathcal{J}} \cap F^{\mathcal{I}} \cap F^{\mathcal{J}} \text{ by induction hypothesis} \\ &= (E \sqcap F)^{\mathcal{I}} \cap (E \sqcap F)^{\mathcal{J}}\end{aligned}$$

For $C = \exists r.E$ the following equations are true since $r^{\mathcal{I}\cap\mathcal{J}} = r^{\mathcal{I}} \cap r^{\mathcal{J}}$:

$$\begin{aligned}(\exists r.E)^{\mathcal{I}\cap\mathcal{J}} &= \{d \in \Delta^{\mathcal{K}} \mid \exists (d,e) \in r^{\mathcal{I}\cap\mathcal{J}}.e \in E^{\mathcal{I}\cap\mathcal{J}}\} \\ &= \{d \in \Delta^{\mathcal{K}} \mid \exists (d,e) \in r^{\mathcal{I}} \cap r^{\mathcal{J}}.e \in E^{\mathcal{I}} \cap E^{\mathcal{J}}\} && (\dagger) \\ &= \{d \in \Delta^{\mathcal{K}} \mid \exists (d,e) \in r^{\mathcal{I}}.e \in E^{\mathcal{I}}\} \cap \{d \in \Delta^{\mathcal{K}} \mid \exists (d,e) \in r^{\mathcal{J}}.e \in E^{\mathcal{J}}\} && (\ddagger) \\ &= (\exists r.E)^{\mathcal{I}} \cap (\exists r.E)^{\mathcal{J}}\end{aligned}$$

The inclusion $\subseteq$ from (†) to (‡) is easy to see while the other direction $\supseteq$ requires the extra conditions of $\mathcal{I}$, $\mathcal{J}$ being standard typicality interpretations and $Qc(C) \subseteq Qc(\mathcal{K})$. Assume $d_F^{\mathcal{U}} \in (\exists r.E)^{\mathcal{I}} \cap (\exists r.E)^{\mathcal{J}}$, then $Qc(C) \subseteq Qc(\mathcal{K})$ implies $d_E^{\emptyset} \in \Delta^{\mathcal{K}}$ and the fact that $\mathcal{I}$, $\mathcal{J}$ are standard implies $(d_F^{\mathcal{U}}, d_E^{\emptyset}) \in r^{\mathcal{I}}$ and $(d_F^{\mathcal{U}}, d_E^{\emptyset}) \in r^{\mathcal{J}}$ as well as $d_E^{\emptyset} \in E^{\mathcal{I}}$ and $d_E^{\emptyset} \in E^{\mathcal{J}}$. It is now easy to see that $(d_F^{\mathcal{U}}, d_E^{\emptyset}) \in r^{\mathcal{I}\cap\mathcal{J}}$ and $d_E^{\emptyset} \in E^{\mathcal{I}\cap\mathcal{J}}$ and thus $d_F^{\mathcal{U}} \in (\exists r.E)^{\mathcal{I}\cap\mathcal{J}}$. □

It follows trivially from Proposition 4.4 that standard typicality interpretations over a shared domain are closed under intersection.

Corollary 4.5. *Let $\mathcal{I} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{I}})$ and $\mathcal{J} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{J}})$ be standard typicality interpretations. Then $\mathcal{I} \cap \mathcal{J}$ is a standard typicality interpretation.*

4.1.1. Defeasible Entailments by Typicality Interpretations

To define the different forms of entailment for the different semantics, we need the notion of a model and characterise under which conditions a typicality interpretation satisfies a DKB. For the classical parts (ABox, TBox), the typicality interpretation should comply with the standard semantics. Concept representatives that are associated with a subset of the DBox, should satisfy the DCIs in that subset.

Definition 4.6 (model of $\mathcal{K}$). Let $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ be a DKB. A typicality interpretation $\mathcal{I} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{I}})$ over $\mathcal{K}$ is a model of $\mathcal{K}$ (written $\mathcal{I} \models \mathcal{K}$) *iff*

1. $\mathcal{I} \models (\mathcal{A}, \mathcal{T})$ and
2. $\mathcal{I}, d_F^{\mathcal{U}} \models \mathcal{U}$ for all $d_F^{\mathcal{U}} \in \Delta^{\mathcal{K}}$.

Definition 4.6 characterises basic conditions under which a DKB is satisfied. In classical semantics, entailments are determined by all models of a given knowledge base. For the non-monotonic semantics that we want to obtain, we use preferred models and impose further restrictions on the set of models of a DKB that are considered to determine entailment. More precisely, we want to investigate the four different semantics, characterised by $(x, y) \in \{\text{rat, rel}\} \times \{\text{prop, nest}\}$. The semantics for the strength of the entailment (i.e., the first component of this pair) are obtained when restricting the set of models of $\mathcal{K}$ to those over *one* specific domain $\Delta_x^{\mathcal{K}}$. The second component imposes further restrictions on those models. For typicality interpretations, we also need to specify when an assertion or a DCI is satisfied, e.g. $\mathcal{I} \models C \sqsubsim D$. Due to the canonical nature of typicality interpretations, entailments (subsumption and instance alike) can be determined by considering single domain elements. Which domain element is selected to decide a subsumption query depends on the underlying typicality domain which in turn depends on the strength of the desired semantics. Thus we obtain a general characterisation of defeasible entailment with recourse to (i) the set of preferred models and (ii) conditions under which a typicality interpretation satisfies a DCI or assertion w.r.t. x. These are made precise later, but we can characterise entailment of a defeasible query by a DKB $\mathcal{K}$ in general. Let

- α be an $\mathcal{EL}_\bot$ expression of the form $C \sqsubsim D$ or $C(a)$ $(a \in sig_I(\mathcal{A}))$,
- $\mathcal{I} \models^x \alpha$ be defined for $x \in \{\text{rat, rel}\}$ and
- the set of preferred models of $\mathcal{K}$: $Mod_{(x,y)}(\mathcal{K})$ that are considered in the decision process is to be defined for $(x, y) \in \{\text{rat, rel}\} \times \{\text{prop, nest}\}$.

Entailment of a defeasible query α by a DKB $\mathcal{K}$ is characterised as

$$\mathcal{K} \models^{(x,y)} \alpha \textit{ iff } \mathcal{I} \models^x \alpha \text{ for all models } \mathcal{I} \in Mod_{(x,y)}(\mathcal{K}). \qquad (\star)$$

A requirement for the preferred models of $\mathcal{K}$ is that $Mod_{(x,y)}$ contains only standard models of $\mathcal{K}$. This restriction is required for all pairs (x, y). Before considering characterisations regarding concrete instantiations of the components x and y of our semantics, we use this initial restriction to standard models and Corollary 4.5 to characterise a single model that is canonical for the set of all standard models over a typicality domain $\Delta^{\mathcal{K}}$. Note that up to this point we considered both, defeasible instance and subsumption checking, simply because the general characterisations of these entailments align so well.

From here we restrict our investigations to subsumption checking, where the ABox has no effect.[7] The main difference between classical canonical models and typicality interpretations is the multitude of concept representatives per concept, which entitles the use of the extended TBox introduced in Definition 3.3 in order to determine materialisation-like entailments based on defeasible information for the appropriate domain elements.

Definition 4.7. Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB, $\Delta^{\mathcal{K}}$ a typicality domain over $\mathcal{K}$, and $\mathcal{U} \subseteq \mathcal{D}$. A typicality interpretation $\mathcal{I} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{I}})$ is a *minimal typicality model* (denoted as $\mathcal{I}(\Delta^{\mathcal{K}})$) if

- it satisfies the property:

$$d_F^{\mathcal{U}} \in \Delta^{\mathcal{K}} \iff F_{\mathcal{U}} \not\sqsubseteq_{\mathcal{T}_{\mathcal{U}}(F)} \bot \qquad (*)$$

- its interpretation mapping satisfies the following conditions for all $d_F^{\mathcal{U}} \in \Delta^{\mathcal{K}}$:
  - $d_F^{\mathcal{U}} \in A^{\mathcal{I}(\Delta^{\mathcal{K}})}$ *iff* $F_{\mathcal{U}} \sqsubseteq_{\mathcal{T}_{\mathcal{U}}(F)} A$, for $A \in sig_C(\mathcal{K})$ and
  - $(d_F^{\mathcal{U}}, d_G^{\emptyset}) \in r^{\mathcal{I}(\Delta^{\mathcal{K}})}$ *iff* $F_{\mathcal{U}} \sqsubseteq_{\mathcal{T}_{\mathcal{U}}(F)} \exists r.G$, for $r \in sig_R(\mathcal{K})$.

Definition 4.7 is general in the sense that no specific typicality domain is fixed, which determines the strength of the semantics in the $x \in \{\text{rat}, \text{rel}\}$ component. We show in the following, that minimal typicality models $\mathcal{I}(\Delta^{\mathcal{K}})$ are canonical for the standard models of $\mathcal{K}$ in the sense that

1. $\mathcal{I}(\Delta^{\mathcal{K}}) \models \mathcal{K}$,
2. $\mathcal{I}(\Delta^{\mathcal{K}})$ is standard, and
3. for all $\mathcal{J} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{J}})$ satisfying 1 and 2, $\mathcal{I}(\Delta^{\mathcal{K}}) \subseteq \mathcal{J}$.

The following intermediary result aligns with the considerations for classical canonical models.

Proposition 4.8. *For a given well–separated DKB $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ and the minimal typicality model $\mathcal{I}(\Delta^{\mathcal{K}})$ for a typicality domain $\Delta^{\mathcal{K}}$, the following holds for all $d_F^{\mathcal{U}} \in \Delta^{\mathcal{K}}$:*

1. $d_F^{\mathcal{U}} \in F_{\mathcal{U}}^{\mathcal{I}(\Delta^{\mathcal{K}})}$

2. $d_F^{\mathcal{U}} \in G^{\mathcal{I}(\Delta^{\mathcal{K}})}$ iff $F_{\mathcal{U}} \sqsubseteq_{\mathcal{T}_{\mathcal{U}}(F)} G$

Proof. Claim 1 is trivial since $F_{\mathcal{U}} \in N_C$ and by Property $(*)$ in Definition 4.7, $d_F^{\mathcal{U}} \in \Delta^{\mathcal{K}} \iff F_{\mathcal{U}} \not\sqsubseteq_{\mathcal{T}_{\mathcal{U}}(F)} \bot$.

We show 2 by induction on the structure of G. The base case, where $G = A$ ($A \in N_C$) follows by Definition 4.7 as $F_{\mathcal{U}} \in N_C$. The cases for $G = \top$ and $G = \bot$ are both trivial. Assume the property holds for two concepts D and E, the case of the induction step where $G = D \sqcap E$ follows quickly from the semantics of conjunction and the induction hypothesis. It remains to show the induction step for $G = \exists r.E$ under the hypothesis $d_X^{\mathcal{U}'} \in E^{\mathcal{I}(\Delta^{\mathcal{K}})} \iff X_{\mathcal{U}'} \sqsubseteq_{\mathcal{T}_{\mathcal{U}'}(X)} E$ for any $d_X^{\mathcal{U}'} \in \Delta^{\mathcal{K}}$. $d_F^{\mathcal{U}} \in G^{\mathcal{I}(\Delta^{\mathcal{K}})}$ implies $\exists d_X^{\emptyset} \in \Delta^{\mathcal{K}}.(d_F^{\mathcal{U}}, d_X^{\emptyset}) \in r^{\mathcal{I}(\Delta^{\mathcal{K}})} \wedge d_X^{\emptyset} \in E^{\mathcal{I}(\Delta^{\mathcal{K}})}$. By Definition 4.7 this implies $F_{\mathcal{U}} \sqsubseteq_{\mathcal{T}_{\mathcal{U}}(F)} \exists r.X$. By IH, $d_X^{\emptyset} \in E^{\mathcal{I}(\Delta^{\mathcal{K}})} \iff X_{\emptyset} \sqsubseteq_{\mathcal{T}_{\emptyset}(X)} E$ and thus, by Proposition 3.4 $X \sqsubseteq_{\mathcal{T}} E$ for a well–separated $\mathcal{K}$. Thus $\mathcal{T} \subseteq \mathcal{T}_{\mathcal{U}}(F)$ implies $F_{\mathcal{U}} \sqsubseteq_{\mathcal{T}_{\mathcal{U}}(F)} \exists r.E$. For the other direction, let $F_{\mathcal{U}} \sqsubseteq_{\mathcal{T}_{\mathcal{U}}(F)} \exists r.E$, 1 directly implies that $d_F^{\mathcal{U}} \in F_{\mathcal{U}}^{\mathcal{I}(\Delta^{\mathcal{K}})}$ and thus $d_F^{\mathcal{U}} \in (\exists r.E)^{\mathcal{I}(\Delta^{\mathcal{K}})} = G^{\mathcal{I}(\Delta^{\mathcal{K}})}$. □

Proposition 4.8 is the main ingredient for showing that a minimal typicality model $\mathcal{I}(\Delta^{\mathcal{K}})$ from Definition 4.7 is in fact a model of the given DKB $\mathcal{K}$ according to Definition 4.6. As a consequence of 1 in Proposition 4.8 and Definition 4.7, it directly follows that $\mathcal{I}(\Delta^{\mathcal{K}})$ is standard.

[7] Once the ABox $\mathcal{A}$ is eventually considered again, all of the definitions and results below apply just the same.

Lemma 4.9. *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB. Then, the minimal typicality model $\mathcal{I}(\Delta^{\mathcal{K}})$ of a typicality domain $\Delta^{\mathcal{K}}$ is a model of $\mathcal{K}$.*

Proof. We need to show that 1 and 2 of Definition 4.6 hold for $\mathcal{I}(\Delta^{\mathcal{K}})$.

1. For all GCIs $G \sqsubseteq H \in \mathcal{T}$ and any $d_F^{\mathcal{U}} \in \Delta^{\mathcal{K}}$, $d_F^{\mathcal{U}} \in G^{\mathcal{I}(\Delta^{\mathcal{K}})}$ *iff* $F_{\mathcal{U}} \sqsubseteq_{\mathcal{T}_{\mathcal{U}}(F)} G$ by Proposition 4.8 and $\mathcal{T} \subseteq \mathcal{T}_{\mathcal{U}}(F)$ implies $F_{\mathcal{U}} \sqsubseteq_{\mathcal{T}_{\mathcal{U}}(F)} H$, which again by Proposition 4.8 holds *iff* $d_F^{\mathcal{U}} \in H^{\mathcal{I}(\Delta^{\mathcal{K}})}$.
2. For 2 of Definition 4.6 we can use a similar argument. For all $d_F^{\mathcal{U}} \in \Delta^{\mathcal{K}}$ and $G \sqsubsim H \in \mathcal{U}$ we need to show $d_F^{\mathcal{U}} \in G^{\mathcal{I}(\Delta^{\mathcal{K}})} \implies d_F^{\mathcal{U}} \in H^{\mathcal{I}(\Delta^{\mathcal{K}})}$. $d_F^{\mathcal{U}} \in G^{\mathcal{I}(\Delta^{\mathcal{K}})}$ is equivalent to $F_{\mathcal{U}} \sqsubseteq_{\mathcal{T}_{\mathcal{U}}(F)} G$ due to Proposition 4.8, which implies $F_{\mathcal{U}} \equiv_{\mathcal{T}_{\mathcal{U}}(F)} F_{\mathcal{U}} \sqcap G$. $G \sqsubsim H \in \mathcal{U}$ implies $F_{\mathcal{U}} \sqcap G \sqsubseteq H \in \mathcal{T}_{\mathcal{U}}(F)$, thus $F_{\mathcal{U}} \sqsubseteq_{\mathcal{T}_{\mathcal{U}}(F)} H$ which is again equivalent to $d_F^{\mathcal{U}} \in H^{\mathcal{I}(\Delta^{\mathcal{K}})}$ by Proposition 4.8. □

Using this result and Prop. 3.4, it is not hard to show that a minimal typicality model, restricted to elements regarding the empty set of DCIs, yields exactly the classical canonical model for the $\mathcal{EL}_{\bot}$ TBox $\mathcal{T}$.

Before we show canonicity of a minimal typicality model w.r.t. standard models of $\mathcal{K}$ over the same domain, we show an intermediary result connecting satisfaction of $\mathcal{K}$ to satisfaction of extended TBoxes $\mathcal{T}_{\mathcal{U}}(F)$.

Proposition 4.10. *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB, F a concept, and $\mathcal{J} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{J}})$ be a standard typicality interpretation over $\mathcal{K}$. Then it holds that*

$$\mathcal{J} \models \mathcal{K} \implies \forall d_F^{\mathcal{U}} \in \Delta^{\mathcal{K}}.\mathcal{J}[F_{\mathcal{U}}/\{d_F^{\mathcal{U}}\}] \models \mathcal{T}_{\mathcal{U}}(F)$$

Proof. Let $\mathcal{J}' = \mathcal{J}[F_{\mathcal{U}}/\{d_F^{\mathcal{U}}\}]$ for simplicity and $d_F^{\mathcal{U}} \in \Delta^{\mathcal{K}}$. Clearly, since $F_{\mathcal{U}} \notin sig(\mathcal{T})$ it holds that $\mathcal{J} \models \mathcal{T}$ implies $\mathcal{J}' \models \mathcal{T}$. Since $\mathcal{J}$ is standard, $d_F^{\mathcal{U}} \in F^{\mathcal{J}}$ holds by Definition 4.3, which implies $\{d_F^{\mathcal{U}}\} = F_{\mathcal{U}}^{\mathcal{J}'} \subseteq F^{\mathcal{J}'}$ (since $F_{\mathcal{U}} \notin sig(F)$). It remains to show for all GCIs $F_{\mathcal{U}} \sqcap G \sqsubseteq H \in \mathcal{T}_{\mathcal{U}}(F) \setminus (\mathcal{T} \cup \{F_{\mathcal{U}} \sqsubseteq F\})$ (i.e. $G \sqsubsim H \in \mathcal{U}$) that $\mathcal{J}' \models F_{\mathcal{U}} \sqcap G \sqsubseteq H$. Since $F_{\mathcal{U}} \notin sig(\mathcal{D})$, it holds that $G^{\mathcal{J}} = G^{\mathcal{J}'}$ and $H^{\mathcal{J}} = H^{\mathcal{J}'}$. Additionally, for all $d_F^{\mathcal{U}} \in \Delta^{\mathcal{K}}$ it holds that $\mathcal{J}, d_F^{\mathcal{U}} \models \mathcal{U}$ ($\mathcal{J} \models \mathcal{K}$), which means for all $G \sqsubsim H \in \mathcal{U}$, that $d_F^{\mathcal{U}} \in G^{\mathcal{J}} \implies d_F^{\mathcal{U}} \in H^{\mathcal{J}}$ (Def. 4.6). Thus, $\{d_F^{\mathcal{U}}\} = F_{\mathcal{U}}^{\mathcal{J}'}$ implies $d_F^{\mathcal{U}} \in G^{\mathcal{J}}$ *iff* $d_F^{\mathcal{U}} \in (F_{\mathcal{U}} \sqcap G)^{\mathcal{J}'}$, it follows that $d_F^{\mathcal{U}} \in (F_{\mathcal{U}} \sqcap G)^{\mathcal{J}'} \implies d_F^{\mathcal{U}} \in H^{\mathcal{J}'}$ which implies $\mathcal{J}' \models F_{\mathcal{U}} \sqcap G \sqsubseteq H$, since no other element than $d_F^{\mathcal{U}}$ may be in the extension of the left-hand side of such a GCI in $\mathcal{T}_{\mathcal{U}}(F)$ under $\mathcal{J}'$. □

Equipped with this result, we show that the minimal typicality model $\mathcal{I}(\Delta^{\mathcal{K}})$ is canonical for the set of all standard models of $\mathcal{K}$ over the domain $\Delta^{\mathcal{K}}$.

Lemma 4.11. *The minimal typicality model $\mathcal{I}(\Delta^{\mathcal{K}})$ is canonical for all standard models $\mathcal{J} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{J}})$ of the DKB $\mathcal{K} = (\mathcal{T}, \mathcal{D})$, i.e. the following properties hold*

1. *$\mathcal{I}(\Delta^{\mathcal{K}}) \models \mathcal{K}$,*
2. *$\mathcal{I}(\Delta^{\mathcal{K}})$ is standard, and*
3. *$\mathcal{I}(\Delta^{\mathcal{K}}) \subseteq \mathcal{J}$ for all standard models $\mathcal{J} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{J}})$ of $\mathcal{K}$.*

Proof. Property 1 is known by Lemma 4.9 and Property 2 follows from Proposition 4.8 and the definition of minimal typicality model (Def. 4.7). To show Property 3, assume for contradiction that there is a standard model $\mathcal{J} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{J}})$ of $\mathcal{K}$ that is not an extension of $\mathcal{I}(\Delta^{\mathcal{K}})$, i.e. $\mathcal{I}(\Delta^{\mathcal{K}}) \not\subseteq \mathcal{J}$. The latter requires that there exists at least one concept or role name, s.t. $A^{\mathcal{I}(\Delta^{\mathcal{K}})} \not\subseteq A^{\mathcal{J}}$ or $r^{\mathcal{I}(\Delta^{\mathcal{K}})} \not\subseteq r^{\mathcal{J}}$, respectively. Hence, we distinguish the following two cases, both leading to contradictions.

Case 1: $\exists A \in sig_C(\mathcal{K}).A^{\mathcal{I}(\Delta^{\mathcal{K}})} \not\subseteq A^{\mathcal{J}}$**.** There must be an element $d_F^{\mathcal{U}} \in A^{\mathcal{I}(\Delta^{\mathcal{K}})} \setminus A^{\mathcal{J}}$. $d_F^{\mathcal{U}} \in A^{\mathcal{I}(\Delta^{\mathcal{K}})}$ implies $F_{\mathcal{U}} \sqsubseteq_{\mathcal{T}_{\mathcal{U}}(F)} A$ by Definition 4.7. By Proposition 4.10 it follows from $\mathcal{J} \models \mathcal{K}$ and $\mathcal{J}$ being standard, that $\mathcal{J}[F_{\mathcal{U}}/\{d_F^{\mathcal{U}}\}] \models \mathcal{T}_{\mathcal{U}}(F)$, which cannot be the case since $\mathcal{J}[F_{\mathcal{U}}/\{d_F^{\mathcal{U}}\}] \not\models F_{\mathcal{U}} \sqsubseteq_{\mathcal{T}_{\mathcal{U}}(F)} A$.

Case 2: $\exists r \in sig_R(\mathcal{K}).r^{\mathcal{I}(\Delta^{\mathcal{K}})} \not\subseteq r^{\mathcal{J}}$**.** There must be a pair $(d_F^{\mathcal{U}}, d_G^{\emptyset}) \in r^{\mathcal{I}(\Delta^{\mathcal{K}})} \setminus r^{\mathcal{J}}$ due to the definition of $\mathcal{I}(\Delta^{\mathcal{K}})$. From Definition 4.7 it follows that $F_{\mathcal{U}} \sqsubseteq_{\mathcal{T}_{\mathcal{U}}(F)} \exists r.G$. Since $\mathcal{J}$ is standard, it follows from Definition 4.3 that $(d_F^{\mathcal{U}}, d_G^{\emptyset}) \notin r^{\mathcal{J}}$ implies $d_F^{\mathcal{U}} \notin (\exists r.G)^{\mathcal{J}}$. Using the same argument as in the previous case, we know that $\mathcal{J}[F_{\mathcal{U}}/\{d_F^{\mathcal{U}}\}] \models \mathcal{T}_{\mathcal{U}}(F)$, which is a contradiction to the assumption because $\mathcal{J}[F_{\mathcal{U}}/\{d_F^{\mathcal{U}}\}] \not\models F_{\mathcal{U}} \sqsubseteq_{\mathcal{T}_{\mathcal{U}}(F)} \exists r.G$ (since $F_{\mathcal{U}} \notin sig(G)$). □

Lemma 4.11 justifies to read facts off from a minimal typicality model and use them as entailments, i.e. as formulas that have to hold for all standard typicality models of a given DKB. We are now ready to obtain concrete semantics for instantiations of our framework, in particular, semantics for $(rat, prop)$ and $(rel, prop)$, simply by fixing specific typicality domains and defining the specific condition for a subsumption to hold in the respective typicality interpretation.

4.2. Propositional Subsumption Entailment

To develop a decision procedure for defeasible subsumption for propositional coverage, we use so-called *"minimal typicality" models* as introduced in the last subsection. The idea is that all role-successor elements are from the level of no typicality (e.g. $d_F^{\emptyset}$) and thus do not need to fulfil any information from the DBox. This realises propositional coverage of defeasible information as this kind of information stays local to role *pre*decessors.

4.2.1. Propositional Relevant Subsumption

To fully instantiate the characterisation of defeasible entailment given in ($\star$) (on page 19), we need to specify the preferred models together with their domain and the conditions under which an interpretation satisfies a GCI or assertion.

Definition 4.12 (Relevant Domain). Let $\mathcal{K}$ be a DKB. The *relevant domain* $\Delta_{\mathrm{rel}}^{\mathcal{K}}$ over the $\mathcal{K}$ is defined as $\Delta_{\mathrm{rel}}^{\mathcal{K}} = \{d_F^{\mathcal{U}} \mid F \in Qc(\mathcal{K}), \mathcal{U} \subseteq \mathcal{D}, F \sqcap \overline{\mathcal{U}} \not\sqsubseteq_{\mathcal{T}} \bot\}$.

Observe that relevant domains have a lattice shape (according to Definition 4.1). By our initial assumption, that all concepts in $Qc(\mathcal{K})$ are consistent with the TBox, it is guaranteed that $\Delta_{\mathrm{rel}}^{\mathcal{K}}$ is a typicality domain according to Definition 4.1. Furthermore, $\Delta_{\mathrm{rel}}^{\mathcal{K}}$ satisfies the Property ($*$) required by Definition 4.7 for a minimal typicality model. The following example illustrates the minimal typicality model for the previously used DKB $\mathcal{K}_{ex1}$.

Example 4.13 (Minimal relevant typicality model). *Consider again the DKB $\mathcal{K}_{ex1}$ from Example 3.2 with the consistent subsets of the DBox $\mathcal{D}_{Worker} = \mathcal{D}_{ex1}$, and $\mathcal{D}_{Boss} = \{Worker \mathrel{\underset{\sim}{\sqsubset}} Productive,\ Boss \mathrel{\underset{\sim}{\sqsubset}} Responsible\}$ w.r.t. Worker and Boss, respectively. The subset-lattice of $\mathcal{D}_{ex1}$ and $\mathcal{I}(\Delta_{rel}^{\mathcal{K}_{ex1}})$ are illustrated in Figure 1 using obvious abbreviations and omitting labels for clarity. Note, that the domain elements are grouped in* grey boxes *according to the subset-lattice indicating which DBox subsets are satisfied by which domain elements.*

According to Definition 4.14, $\mathcal{I}(\Delta_{rel}^{\mathcal{K}_{ex1}}) \models Worker \mathrel{\underset{\sim}{\sqsubset}} \exists superior.Boss$, as well as $\mathcal{I}(\Delta_{rel}^{\mathcal{K}_{ex1}}) \models Boss \mathrel{\underset{\sim}{\sqsubset}} Responsible \sqcap Productive$, because $d_{Worker}^{\mathcal{D}_{Worker}}$ and $d_{Boss}^{\mathcal{D}_{Boss}}$ satisfy $\mathcal{D}_{Worker}$ and $\mathcal{D}_{Boss}$, respectively.

As the second component in specifying concrete semantics, we need to determine the conditions for $\mathcal{I} \models^x C \mathrel{\underset{\sim}{\sqsubset}} D$. Since minimal typical modes are canonical models, one needs to find the representative element in $\Delta_{\mathrm{rel}}^{\mathcal{K}}$, that is most typical w.r.t. the concept it represents and thus fulfilling as much defeasible information as possible. In general the most typical concept representative element $d_F^{\mathcal{U}}$ of a concept F cannot be determined by the maximal cardinality of $\mathcal{U}$ for all representatives of F in $\Delta_{\mathrm{rel}}^{\mathcal{K}}$. To clarify this, consider the relevant domain $\Delta_{\mathrm{rel}}^{\mathcal{K}}$ that contains two elements $d_F^{\mathcal{U}_1}$ and $d_F^{\mathcal{U}_2}$ with $|\mathcal{U}_1| = |\mathcal{U}_2|, \mathcal{U}_1 \neq \mathcal{U}_2$ and no element $d_F^{\mathcal{U}}$ with $|\mathcal{U}| > |\mathcal{U}_1|$ is contained in $\Delta_{\mathrm{rel}}^{\mathcal{K}}$. In this scenario, it is not clear which of the DBox subsets that are consistent with F and maximal by cardinality are to be used for deciding subsumption. In order to align our semantics with the entailments contained in minimal relevant closure in [8], the most typical representative of a concept F is defined as $d_F^{\mathcal{D}_F}$, where $\mathcal{D}_F$ is determined by means of justification (see Subsection 3.1 on page 10). Conditions for a typicality interpretation over the relevant domain to satisfy a defeasible subsumption is then characterised as follows.

Definition 4.14 (Defeasible subsumption under relevant strength, $\models^{rel}$). Let $\mathcal{I}$ be a typicality interpretation over the relevant domain $\Delta^{\mathcal{K}}_{\text{rel}}$. Then $\mathcal{I}$ *satisfies a defeasible subsumption* $C \mathrel{\underset{\sim}{\sqsubset}} D$ (written $\mathcal{I} \models^{rel} C \mathrel{\underset{\sim}{\sqsubset}} D$) *iff* $d_C^{\mathcal{D}_C} \in D^{\mathcal{I}}$.

Note that the domain element chosen in Definition 4.14 (likewise in Definition 4.19) to determine satisfaction of a defeasible subsumption, determines, together with the shape of the underlying typicality domain, the strength of the resulting semantics. The coverage of the resulting semantics is determined by the set of models of $\mathcal{K}$ that are considered to decide defeasible entailments. We can now define entailment under the semantics characterised by $(rel, prop)$ in accordance to the general entailment characterisation in $(\star)$.

Definition 4.15. Let $\mathcal{K}$ be a DKB. The typicality interpretations $\mathcal{I} \in Mod_{(rel,prop)}$ that are considered in defining entailments based on propositional relevant semantics satisfy the following properties:

1. $\mathcal{I}$ is standard,
2. $\mathcal{I}$ uses the relevant domain $\Delta^{\mathcal{K}}_{\text{rel}}$.

A defeasible knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ entails a defeasible subsumption $C \mathrel{\underset{\sim}{\sqsubset}} D$ under propositional relevant semantics (written $\mathcal{K} \models^{(rel,prop)} C \mathrel{\underset{\sim}{\sqsubset}} D$) *iff* $\mathcal{I} \models^{rel} C \mathrel{\underset{\sim}{\sqsubset}} D$ for all $\mathcal{I} \in Mod_{(rel,prop)}(\mathcal{K})$.

As a consequence of Lemma 4.11, we can characterise propositional relevant entailment of subsumption using the minimal typicality model over the relevant domain $\Delta^{\mathcal{K}}_{\text{rel}}$, i.e.

$$\mathcal{K} \models^{(rel,prop)} C \mathrel{\underset{\sim}{\sqsubset}} D \textit{ iff } \mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}}) \models^{rel} C \mathrel{\underset{\sim}{\sqsubset}} D.$$

We are now ready to show equivalence of materialisation-based reasoning as described in [8], restricted to the DL $\mathcal{EL}_\bot$ and propositional relevant semantics given by standard typicality models.

Theorem 4.16. *Let $\mathcal{K}$ be an $\mathcal{EL}_\bot$ DKB. Propositional relevant entailment coincides with materialisation-based relevant entailment, i.e.*

$$\mathcal{K} \models^{(rel,prop)} C \mathrel{\underset{\sim}{\sqsubset}} D \iff \mathcal{K} \models^{(rel,mat)} C \mathrel{\underset{\sim}{\sqsubset}} D.$$

Proof. From Definition 4.15 and Lemma 4.11 we know that $\mathcal{K} \models^{(rel,prop)} C \mathrel{\underset{\sim}{\sqsubset}} D$ can be decided by $\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}})$, i.e. $d_C^{\mathcal{D}_C} \in D^{\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}})}$. Proposition 4.8 shows that this is equivalent to deciding $C_{\mathcal{D}_C} \sqsubseteq_{\mathcal{T}_{\mathcal{D}_C}(C)} D$. From Lemma 3.5 this is in turn equivalent to $\overline{\mathcal{D}_C} \sqcap C \sqsubseteq_{\mathcal{T}} D$, which is precisely the definition of $\mathcal{K} \models^{(rel,mat)} C \mathrel{\underset{\sim}{\sqsubset}} D$.

A direct consequence and main motivation for this result, is that the same KLM postulates as discussed in [8] also hold for propositional relevant entailment. As we show next, this is even the case for the inferentially weaker propositional rational semantics.

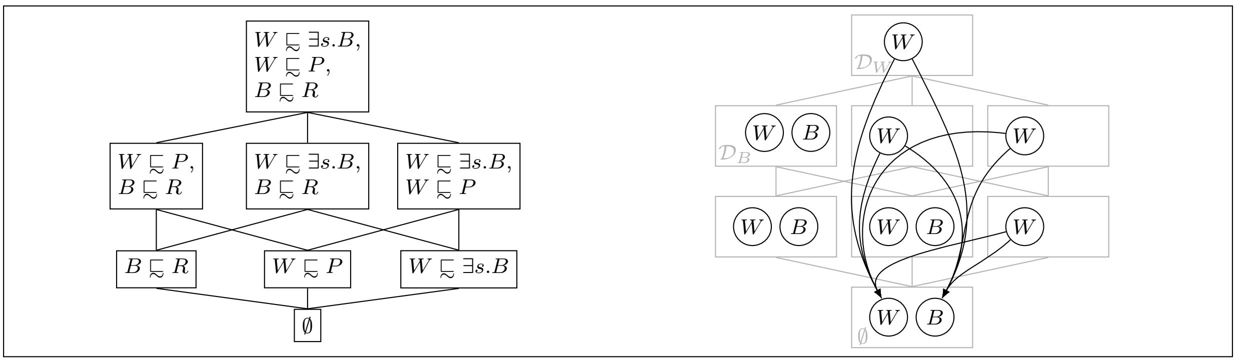

Figure 1: a Subset lattice of $\mathcal{D}_{ex1}$ and **b** $\mathcal{I}(\Delta^{\mathcal{K}_{ex1}}_{rel})$

4.2.2. Propositional Rational Subsumption

Despite being inferentially weaker than their relevant counter-part, rational entailment is of interest mainly for two reasons. First, rational entailment can be considered more well-behaved than relevant closure in terms of KLM postulates they satisfy. The postulates Disjunction (Or), Cautious Monotonicity (CM) and Rational Monotonicity (RM) are satisfied by materialisation-based rational entailment but not by materialisation-based relevant entailment. Second, as it will turn out in our complexity analysis, the nested coverage semantics come at the cost of an extra non-deterministic guessing step. Therefore, reasoning w.r.t. nested rational semantics could be more feasible in practice than reasoning under nested relevant semantics.

We proceed in this section as for the relevant semantics before and define a typicality domain whose granularity of sets of defeasible information is coarser and allows to obtain (only) rational strength.

Definition 4.17 (Rational Domain). Let $\mathcal{K}$ be a DKB . The *rational domain* $\Delta^{\mathcal{K}}_{\text{rat}}$ over the DKB $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ with *partition*$(\mathcal{D}) = (E_0, \ldots, E_n)$ and $\mathcal{D}_i = \bigcup_{j=i}^{n} E_j$ $(0 \leq i \leq n)$ is defined as

$$\Delta^{\mathcal{K}}_{\text{rat}} = \{d^{\mathcal{D}_i}_F \mid F \in Qc(\mathcal{K}), F \sqcap \overline{\mathcal{D}_i} \not\sqsubseteq_{\mathcal{T}} \bot, 0 \leq i \leq n\}.$$

By the initial assumption, that all concepts in $Qc(\mathcal{K})$ are consistent with the TBox, it follows that $\Delta^{\mathcal{K}}_{\text{rat}}$ is a typicality interpretation according to Definition 4.1 and due to the total order of the $\mathcal{D}_i$ w.r.t. $\subset$, it has a chain shape. The size of the rational domain is polynomial in the size of the input DKB $\mathcal{K}$. The following example showcases the shape of the rational typicality domain (similar to the presentation of the lattice domain in Figure 1) as well as the effect of inheritance blocking.

Example 4.18 (Minimal rational typicality model). *Consider again the DKB $\mathcal{K}_{ex1}$ from Example 3.2 with partition$(\mathcal{D}_{ex1}) = \{E_1, E_2, E_3\}$, leading to the chain of represented DBox subsets ($\Gamma(\Delta^{\mathcal{K}_{ex1}}_{rat})$) $\mathcal{D}_1 = \mathcal{D}_{ex1}$, $\mathcal{D}_2 = \{Boss \mathrel{\underset{\sim}{\sqsubset}} Responsible\}$ and $\mathcal{D}_3 = \emptyset$. The DBox partition and $\mathcal{I}(\Delta^{\mathcal{K}_{ex1}}_{rat})$ are illustrated in Figure 2 using obvious abbreviations. Note, that the domain elements are grouped in* grey boxes *according to the DBox subset chain indicating which DBox subsets are satisfied by which domain elements.*

According to Definition 4.19, $\mathcal{I}(\Delta^{\mathcal{K}_{ex1}}_{rat}) \models Worker \mathrel{\underset{\sim}{\sqsubset}} \exists superior.Boss$, as well as $\mathcal{I}(\Delta^{\mathcal{K}_{ex1}}_{rat}) \models Boss \mathrel{\underset{\sim}{\sqsubset}} Responsible$. However, as opposed to the minimal typicality model over the relevant domain, the property Productive (from $Worker \mathrel{\underset{\sim}{\sqsubset}} Productive$) is not satisfied for the element $d^{\mathcal{D}_2}_{Boss}$. This shows once more how the rough granularity of represented DBox subsets makes the minimal typicality model over the rational domain subject to inheritance blocking.

Hence, the representative domain elements of the same concept F are also totally ordered according to the DBox subset $\mathcal{D}_i$ they satisfy. In rational semantics it is therefore obvious which of the representatives is the most typical one and is to be used to read off the information to answer the query. The condition for a typicality interpretation over the rational domain to satisfy a defeasible subsumption is then characterised as follows. Recall that this condition determines only the strength of the resulting semantics.

Definition 4.19 (Defeasible subsumption under rational strength, $\models^{rat}$).
Let $\mathcal{K}$ be a DKB and $\mathcal{I}$ be a typicality interpretation over $\Delta^{\mathcal{K}}_{\text{rat}}$. Then $\mathcal{I}$ *satisfies a defeasible subsumption* $C \mathrel{\underset{\sim}{\sqsubset}} D$ (written $\mathcal{I} \models^{rat} C \mathrel{\underset{\sim}{\sqsubset}} D$) *iff* $d^{\mathcal{D}_i}_C \in D^{\mathcal{I}}$ for the smallest i s.t. $d^{\mathcal{D}_i}_C \in \Delta^{\mathcal{K}}_{\text{rat}}$.

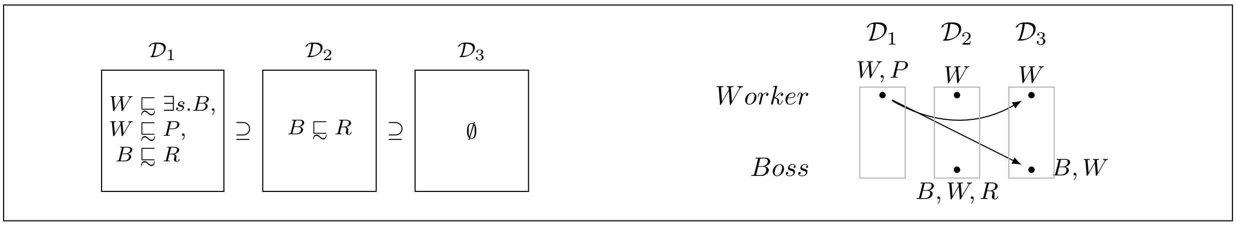

Figure 2: **a** Partition *partition*$(\mathcal{D}_{ex1})$ and **b** $\mathcal{I}(\Delta^{\mathcal{K}_{ex1}}_{rel})$

The definition for entailment of a defeasible subsumption by a DKB $\mathcal{K}$ w.r.t. propositional rational semantics is analogous to the relevant case as well.

Definition 4.20. The typicality interpretations $\mathcal{I} \in Mod_{(rat,prop)}$ that are considered in defining entailments based on propositional rational semantics satisfy the following properties:

1. $\mathcal{I}$ is standard,
2. $\mathcal{I}$ is defined over $\Delta^{\mathcal{K}}_{\text{rat}}$.

A defeasible knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ entails a defeasible subsumption $C \sqsubsetsim D$ under propositional rational semantics (written $\mathcal{K} \models^{(rat,prop)} C \sqsubsetsim D$) *iff* $\mathcal{I} \models^{rat} C \sqsubsetsim D$ for all $\mathcal{I} \in Mod_{(rat,prop)}(\mathcal{K})$.

We can show the equivalence between materialisation-based and propositional rational entailment.

Theorem 4.21. *Let $\mathcal{K}$ be an $\mathcal{EL}_\bot$ DKB. Propositional rational entailment coincides with materialisation-based rational entailment, i.e.*

$$\mathcal{K} \models^{(rat,prop)} C \sqsubsetsim D \Longleftrightarrow \mathcal{K} \models^{(rat,mat)} C \sqsubsetsim D.$$

Proof. From Definition 4.19 and Lemma 4.11 we know that $\mathcal{K} \models^{(rat,prop)} C \sqsubsetsim D$ can be decided by $\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rat}})$, i.e. $d_C^{\mathcal{D}_C} \in D^{\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rat}})}$. Proposition 4.8 shows that this is equivalent to deciding $C_{\mathcal{D}_C} \sqsubseteq_{\mathcal{T}_{\mathcal{D}_C}(C)} D$. By Lemma 3.5 this is in turn equivalent to $\overline{\mathcal{D}_C} \sqcap C \sqsubseteq_{\mathcal{T}} D$, which is the definition of $\mathcal{K} \models^{(rel,mat)} C \sqsubsetsim D$.

Even though we characterised two semantics by defining specific interpretation domains, we can capture a lot of the results in the remaining paper for general typicality domains. Naturally, those general results apply to the specific rational and relevant domain just the same.

We continue now to use the typicality interpretation formalism to resolve the criticism of ignoring defeasible information for quantified concepts.

4.3. Nested Subsumption Entailment

The main difference between propositional coverage and nested coverage is clearly the treatment of defeasible information for the role-successors. While propositional coverage is deliberately oblivious of defeasible information for role-successors, nested coverage tries to use as much defeasible information for role-successors as preservation of consistency admits. We pursue this goal by characterising the set of maximal typicality models, where each role successor required by $\mathcal{K}$ is chosen such that it satisfies a subset of DCIs from $\mathcal{D}$ that is of maximal cardinality while not causing an inconsistency. This provides us with a restriction on the models of $\mathcal{K}$ (those of maximal typicality) that are considered to decide subsumption queries of the kind: $\mathcal{K} \models^{(x,nest)} C \sqsubsetsim D$. Role edges whose end point is a representative domain element for a bigger subset of $\mathcal{D}$ (than other end points), are considered more typical. Intuitively, we can obtain typicality models of maximal typicality, by extending non-maximal typicality models with more typical role edges.

The crucial step in transforming a typicality model into a maximal typicality model is to find the role-successor of maximal typicality. Suppose, a typicality model has $d_E^{\emptyset}$ as r-successor and we want to find the maximal typical role-successor caused by the existential restriction $\exists r.E$. Now, in order to upgrade the typicality of the E-representative, an r-edge to an E-representative that satisfies more DCIs, i.e., a bigger subset of $\mathcal{D}$, is introduced.[8] Obtaining a more typical interpretation by introducing such an edge is called *upgrading the typicality* of an interpretation. If we consider from the standard typicality models of a DKB $\mathcal{K}$ only those models that are more typical w.r.t. some set of upgraded role edges, we can potentially obtain more entailments and in particular those that are missing in propositional and materialisation-based defeasible entailment relations. Note, that for some upgrades all standard models realising this upgrade

[8]The "old" r-edge can remain as it does not affect reasoning.

contain new role edges, e.g. $(d_F^{\mathcal{U}}, d_G^{\emptyset})$. In our characterisation of maximal typicality, we maximise those new edges as well.

We implement this idea by a two-step fixpoint construction. Initially, we start from the minimal typicality model and determine its typicality upgrades in the first step. In general, upgrades of a (minimal) typicality model are not necessarily models of $\mathcal{K}$. However, they can either be completed into models of $\mathcal{K}$ or they cannot, since by the new edge, some disjointness constraint in the TBox can be violated. If a typicality upgrade can be completed into models of $\mathcal{K}$, it can be done so in a minimal way which is the second step. Since this model completion could be required to contain role edges that have not been present in the minimal typicality model, we need to iteratively keep upgrading and (model) completing until none of the potential upgrades admit a model completion anymore. At this point, maximal typicality is reached. As it happens, one typicality upgrade (of a certain role-edge) may block another upgrade, i.e. in the presence of both upgrades no model completion is possible. In this case we need to consider further upgrades of both typicality models, each containing only one of the two upgraded edges. This shows that it is possible to reach a variety of maximal typicality models during this iterative upgrade procedure, starting from the minimal typicality model.

Conceptually, this procedure corresponds to an iterative restriction of the set of standard models that are considered to decide defeasible subsumption. Initially, all edges that occur in every standard model ($Mod_{(x,prop)}$) of $\mathcal{K}$ are required to be maximal w.r.t. their typicality, i.e. those occurring in $\mathcal{I}(\Delta_x^{\mathcal{K}})$. New role edges that are induced by this maximality requirement, are again required to be of maximal typicality, hence the iterative nature of this process. Eventually, we are able to characterise a restriction on the standard typicality models of a DKB $\mathcal{K}$, i.e. $Mod_{(x,nest)}(\mathcal{K}) \subseteq Mod_{(x,prop)}(\mathcal{K})$, where $Mod_{(x,nest)}(\mathcal{K})$ includes only those models that are maximal w.r.t. the typicality of the iteratively determined induced role-successors and the role-successors occurring in all standard models of $\mathcal{K}$.

We begin to develop the two-step fixpoint construction by characterising the upgrades that a typicality interpretation allows. As before, we remain as general as possible at first, w.r.t. the underlying domain and considering an ABox, before specifically looking at subsumption and eventually at instance checking in Section 5.

Definition 4.22. Let $\mathcal{I} = (\Delta^{\mathcal{K},\mathcal{I}}, \cdot^{\mathcal{I}})$ and $\mathcal{J} = (\Delta^{\mathcal{K},\mathcal{J}}, \cdot^{\mathcal{J}})$ be typicality interpretations for $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$. The *set of more typical role edges* for a given role r in $\mathcal{I}$ is defined as

$$TR_{\mathcal{I}}(r) = \{(d, d_H^{\mathcal{U}'}) \in \Delta^{\mathcal{K},\mathcal{I}} \times \Delta^{\mathcal{K},\mathcal{I}} \setminus r^{\mathcal{I}} \mid \exists \mathcal{U} \subseteq \mathcal{D}.(d, d_H^{\mathcal{U}}) \in r^{\mathcal{I}} \wedge \mathcal{U} \subset \mathcal{U}' \subseteq \mathcal{D}_H\}.$$

$\mathcal{J}$ is a *typicality extension* of $\mathcal{I}$ *iff*

1. $\Delta^{\mathcal{K},\mathcal{J}} = \Delta^{\mathcal{K},\mathcal{I}}$,
2. $A^{\mathcal{J}} = A^{\mathcal{I}}$ (for $A \in N_C$),
3. $a^{\mathcal{J}} = a^{\mathcal{I}}$ (for $a \in sig_I(\mathcal{A})$)
4. $r^{\mathcal{J}} = r^{\mathcal{I}} \cup R$, where $R \subseteq TR_{\mathcal{I}}(r)$ (for $r \in sig_R(\mathcal{K})$), and
5. $\exists r \in sig_R(\mathcal{K}).\ r^{\mathcal{I}} \subset r^{\mathcal{J}}$.

The *set of all typicality extensions* of a typicality interpretation $\mathcal{I}$ is *typ*$(\mathcal{I})$.

Note that the starting points of the edges contained in $TR_{\mathcal{I}}(r)$ can be concept or individual representatives. Consider the DKB $\mathcal{K}_{ex1}$ from Example 3.2 and the minimal typicality model $\mathcal{I}(\Delta_{rel}^{\mathcal{K}_{ex1}})$ (Figure 1). From Example 4.13, one can see that $d_{Worker}^{\mathcal{D}_{Worker}} \notin (\exists superior.Responsible)^{\mathcal{I}(\Delta_{rel}^{\mathcal{K}_{ex1}})}$. Due to the edge $(d_{Worker}^{\mathcal{D}_{Worker}}, d_{Boss}^{\emptyset})$ in $superior^{\mathcal{I}(\Delta_{rel}^{\mathcal{K}_{ex1}})}$, it holds that $(d_{Worker}^{\mathcal{D}_{Worker}}, d_{Boss}^{\mathcal{D}_{Boss}}) \in TR_{\mathcal{I}(\Delta_{rel}^{\mathcal{K}_{ex1}})}(superior)$ (Def. 4.22). By extending the minimal typicality model to contain this upgrade (let $\mathcal{J} = \mathcal{I}(\Delta_{rel}^{\mathcal{K}_{ex1}})[superior/superior^{\mathcal{I}(\Delta_{rel}^{\mathcal{K}_{ex1}})} \cup \{(d_{Worker}^{\mathcal{D}_{Worker}}, d_{Boss}^{\mathcal{D}_{Boss}})\}]$) we are able to conclude $d_{Worker}^{\mathcal{D}_{Worker}} \in (\exists superior.Responsible)^{\mathcal{J}}$. However, suppose the extra GCI $\exists superior.Responsible \sqsubseteq \exists coworker.Worker$ belongs to the TBox. As a consequence, the interpretation $\mathcal{J}$ would not satisfy the TBox anymore, because $d_{Worker}^{\mathcal{D}_{Worker}}$ has no *coworker* successors. This showcases how a typicality upgrade by Definition 4.22 need not satisfy the underlying DKB. Because of such situations, a completion of an upgraded interpretation may become necessary to make it satisfy the knowledge base again. The set of all extensions of an upgraded typicality interpretation that are standard models of $\mathcal{K}$ is exactly the subclass of $Mod_{(x,prop)}(\mathcal{K})$ where the particular upgrade is contained in all models of that class.

Definition 4.23. Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB and $\Delta^{\mathcal{K}}$ a typicality domain over $\mathcal{K}$. A typicality interpretation $\mathcal{I} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{I}})$ is a *model completion* of a typicality interpretation $\mathcal{J} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{J}})$ *iff*

1. $\mathcal{J} \subseteq \mathcal{I}$,
2. $\mathcal{I} \models \mathcal{K}$, and
3. $\mathcal{I}$ is standard

The *set of all model completions* of $\mathcal{J}$ is denoted as $mc(\mathcal{J})$.

Note that $mc(\mathcal{J})$ can also be empty. An interpretation that is a model completion to itself is called a *safe model* and obviously satisfies the properties of Definition 4.23. So, for any typicality interpretation $\mathcal{J}$, all interpretations in $mc(\mathcal{J})$ are safe models. Observe, that it is no restriction to consider only model completions of $\mathcal{I}$ that belong to $mc(\mathcal{I})$, since if $mc(\mathcal{I}) = \emptyset$ then no extension of $\mathcal{I}$ will be a model of $\mathcal{K}$.

Naturally, the model completions of a typicality upgrade may introduce new edges. Those edges that appear in all model completions of a typicality upgrade, are considered necessary w.r.t. the typicality upgrade, and thus we want to include them in the characterisation of maximal typicality as well. We show that the model completions of an upgraded typicality interpretation are closed under intersection. This naturally allows to characterise a *minimal* model completion that is equivalent to the intersection of all model completions. The iterative upgrade procedure will then proceed to increase the typicality of this minimal model completion, rather than considering upgrades for all model completions.[9]

Proposition 4.24. *For a typicality interpretation $\mathcal{I} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{I}})$ over the DKB $\mathcal{K} = (\mathcal{T}, \mathcal{D})$, the set of model completions $mc(\mathcal{I})$ is closed under intersection.*

Proof. Since $mc(\mathcal{I})$ are considered for finite domains $\Delta^{\mathcal{K}}$ and it is only necessary to consider the signature of $\mathcal{K}$, it suffices for our purposes to show closure under finite intersection. Thus, we show for two interpretations $\mathcal{J}_1, \mathcal{J}_2 \in mc(\mathcal{I})$ that $(\mathcal{J}_1 \cap \mathcal{J}_2) \in mc(\mathcal{I})$ holds, i.e. all three conditions of model completions (in Definition 4.23) hold for $\mathcal{J}_1 \cap \mathcal{J}_2$. Condition 1 follows quickly from the extension of set inclusion to interpretations over the same domain. $\mathcal{I} \subseteq \mathcal{J}_1$ and $\mathcal{I} \subseteq \mathcal{J}_2$ imply $\mathcal{I} \subseteq \mathcal{J}_1 \cap \mathcal{J}_2$. Condition 2 follows from Proposition 4.4 Claim 2. $\mathcal{J}_i \models \mathcal{T}$ implies $C^{\mathcal{J}_i} \subseteq D^{\mathcal{J}_i}$ for all $C \sqsubseteq D \in \mathcal{T}$ for both $i \in \{1, 2\}$ and by Proposition 4.4 Claim 2 the same holds for $\mathcal{J}_1 \cap \mathcal{J}_2$, thus $\mathcal{J}_1 \cap \mathcal{J}_2 \models \mathcal{T}$. Condition 2 in the Definition of models is equivalent to

$$G \mathrel{\underset{\sim}{\sqsubset}} H \in \mathcal{U} \implies G^{\mathcal{J}} \cap \{d_F^{\mathcal{X}} \in \Delta^{\mathcal{K}} \mid \mathcal{X} = \mathcal{U}\} \subseteq H^{\mathcal{J}} \cap \{d_F^{\mathcal{X}} \in \Delta^{\mathcal{K}} \mid \mathcal{X} = \mathcal{U}\}. \qquad (**)$$

for $\mathcal{U} \subseteq \mathcal{D}$ and $(**)$ holds for both $\mathcal{J}_1$ and $\mathcal{J}_2$. This way, it is not hard to see that, as before, Proposition 4.4 implies $(\mathcal{J}_1 \cap \mathcal{J}_2), d_F^{\mathcal{U}} \models \mathcal{U}$ for all $F \in Qc(\mathcal{K})$ and $\mathcal{U} \subseteq \mathcal{D}$.

Condition 3 is satisfied by Corollary 4.5. □

Proposition 4.24 implies that if there is a model completion for a typicality interpretation $\mathcal{J}$, then there is also a unique model completion of $\mathcal{J}$ that is minimal in the sense that it stems from a minimal number of extensions.

[9] This strategy removes an effect that earlier versions of the typicality model approach were still subject to, where some typicality upgrades could be missed because they were blocked by arbitrary information that can be satisfied when continuing to upgrade all model completions. Such arbitrary information cannot exist in the minimal model completion.

Definition 4.25. The *minimal model completion*, of a typicality interpretation $\mathcal{J}$ with $mc(\mathcal{J}) \neq \emptyset$ is defined as

$$mmc(\mathcal{J}) = \bigcap_{\mathcal{I} \in mc(\mathcal{J})} \mathcal{I}.$$

Recall that the minimal model completion of a typicality extension need not be maximal w.r.t. the typicality of role successors, as it may introduce new edges that require further typicality upgrades. We characterise maximal typicality by typicality extensibility. A typicality interpretation $\mathcal{I}$ is said to be typicality extensible, if there exists a typicality upgrade $\mathcal{J}$ in $typ(\mathcal{I})$ such that $mc(\mathcal{J}) \neq \emptyset$. Hence, a typicality interpretation is a maximal typicality interpretation, if it is not typicality extensible. The overall typicality maximisation procedure iteratively performs typicality upgrades and model completions until reaching maximal typicality. To formalise this process, we introduce some notation and an upgrade operator.

Definition 4.26. The *set of all safe models* $P(\Delta^{\mathcal{K}})$ of a typicality domain $\Delta^{\mathcal{K}}$ over a DKB $\mathcal{K}$ is

$$P(\Delta^{\mathcal{K}}) = \{\mathcal{J} \mid \mathcal{J} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{J}}) \wedge \mathcal{J} \in mc(\mathcal{J})\}.$$

The *typicality upgrade operator* $T : 2^{P(\Delta^{\mathcal{K}})} \rightarrow 2^{P(\Delta^{\mathcal{K}})}$ is defined for $S \subseteq P(\Delta^{\mathcal{K}})$ as:

1. $T(S) = S \setminus \{\mathcal{I}\} \cup \{mmc(\mathcal{J}) \mid \mathcal{J} \in typ(\mathcal{I}) \wedge mc(\mathcal{J}) \neq \emptyset\}$, if $\mathcal{I} \in S$ is typicality extensible,
2. $T(S) = S$, otherwise.

For a given set of model completions $S \subseteq P(\Delta^{\mathcal{K}})$, the *fixpoint of* T is $T_m(S)$ if $T_m(S) = T_{m+1}(S)$ with

- $T_0(S) = S$ and
- $T_i(S) = T(T_{i-1}(S))$ $(i > 0)$.

The *set of maximal typicality extensions* of the typicality models in S is $typ^{\mathsf{max}}(S) = T_m(S)$.

Note, that $P(\Delta^{\mathcal{K}})$ is finite, when considering only the finite signature given by the DKB $\mathcal{K}$ and finite typicality domains $\Delta^{\mathcal{K}}$. Typicality upgrades may block each other due to disjointness constraints in the TBox. The following example illustrates how such a typicality extension can lead to multiple different maximal typicality interpretations, starting from a single interpretation.

Example 4.27. *We extend the DKB from Example 4.13 to DKB* $\mathcal{K}_{ex2} = (\mathcal{T}_{ex2}, \mathcal{D}_{ex1})$ *with the TBox*

$$\mathcal{T}_{ex2} = \mathcal{T}_{ex1} \cup \{\exists superior.\exists superior.Responsible \sqsubseteq \bot\}.$$

Let the role edge $(d^{\mathcal{D}}_{Worker}, d^{\emptyset}_{Worker}) \in superior^{\mathcal{I}(\Delta^{\mathcal{K}_{ex2}}_{rel})}$ *be upgraded to* $(d^{\mathcal{D}}_{Worker}, d^{\mathcal{D}}_{Worker})$ *and likewise* $(d^{\mathcal{D}}_{Worker}, d^{\emptyset}_{Boss}) \in superior^{\mathcal{I}(\Delta^{\mathcal{K}_{ex2}}_{rel})}$ *to* $(d^{\mathcal{D}}_{Worker}, d^{\mathcal{D}_{Boss}}_{Boss})$*. If both of these upgrades exist in the same typicality extension* $\mathcal{J}$*, it does not admit to a model completion, as an inconsistency would be caused by*

$$d^{\mathcal{D}}_{Worker} \in (\exists superior.\exists superior.Responsible)^{\mathcal{J}}.$$

The typicality upgrade $(d^{\mathcal{D}}_{Worker}, d^{\{Worker \sqsubseteq\!\!\sim Productive\}}_{Boss})$*, however, is "allowed" to occur in a typicality extension, leading to the entailment of* $Worker \sqsubseteq\!\!\sim \exists superior.(Boss \sqcap Productive)$*. This shows that inheritance blocking can be remedied even for quantified concepts when upgrading typicality of successors in a lattice domain, as the granularity of considered DBox subsets, allows to "individually disregard" contradicting DCIs. In the rational domain, only an even less typical (not satisfying* $Worker \sqsubseteq\!\!\sim Productive$*) Boss representative would be allowed as a superior successor of a* $Worker$ *representative.*

The different maximal typicality interpretations are the reason why the iterative T-operator has to consider sets of typicality interpretations S instead of single interpretations. Since we want to characterise semantics that are usually described as *cautious*, we want to maintain all mutually exclusive upgrade sequences and eventually end up with a variety of maximal typicality models. How does this correspond to restricting the set of standard typicality models of a DKB $\mathcal{K}$? Each of these maximal typicality models characterises a subset of all standard models of $\mathcal{K}$ for which a particular (maximal) set of typicality upgrades is satisfied. For cautious semantics we want to reason over the union of those restrictions of $Mod_{(x,prop)}(\mathcal{K})$. Such entailments can be characterised by considering only the set of all maximal typicality models, starting from a minimal typicality model $\mathcal{I}(\Delta^{\mathcal{K}})$, for $Mod_{(x,nest)}(\mathcal{K})$.

4.3.1. Nested Relevant Subsumption

Most of the prerequisites for the semantic characterisation of nested relevant entailment were introduced already in a general way. We consider as a foundation again the relevant domain $\Delta^{\mathcal{K}}_{\text{rel}}$ from Definition 4.12 and hence the entailment $\models^{rel}$ as introduced in Definition 4.14. The set of considered models for nested relevant entailment coincide with the maximal typicality models obtained from the minimal typicality model over the relevant domain.

Definition 4.28 (Defeasible subsumption under nested relevant semantics, $\models^{(rel,nest)}$). Let $Mod_{(rel,nest)}(\mathcal{K}) = typ^{\mathsf{max}}(\{\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}})\})$. A defeasible knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ entails a defeasible subsumption $C \sqsubseteq\!\!\!\!\sim D$ under nested relevant semantics (written $\mathcal{K} \models^{(rel,nest)} C \sqsubseteq\!\!\!\!\sim D$) *iff* $\mathcal{I} \models^{rel} C \sqsubseteq\!\!\!\!\sim D$ for all $\mathcal{I} \in Mod_{(rel,nest)}(\mathcal{K})$.

Our main goal of this article is to develop a reasoning method that computes defeasible subsumptions such that typicality of objects described by nested existential restrictions is regarded. Our claim is now that reasoning under $(rel, nest)$ semantics does remedy the short-coming of materialisation-based reasoning which does not regard typicality of objects described by nested existential restrictions. So, it remains to show that reasoning with maximal typicality models supports strictly more entailments than reasoning with all typicality models, hence materialisation-based reasoning—in particular when it comes to role successors.

Theorem 4.29. *For two $\mathcal{EL}_\bot$ concepts C, D and an $\mathcal{EL}_\bot$ DKB $\mathcal{K}$ the following holds:*

1. $\mathcal{K} \models^{(rel,mat)} C \sqsubseteq\!\!\!\!\sim D \implies \mathcal{K} \models^{(rel,nest)} C \sqsubseteq\!\!\!\!\sim D$*, and*
2. $\mathcal{K} \models^{(rel,mat)} C \sqsubseteq\!\!\!\!\sim D \not\Longleftarrow \mathcal{K} \models^{(rel,nest)} C \sqsubseteq\!\!\!\!\sim D$

Proof. Claim 1 follows from the fact that the minimal typicality model $\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}})$ is included (according to Definition 2.2) in all maximal typicality models of $\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}})$, i.e. $\mathcal{J} \in typ^{\mathsf{max}}(\{\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}})\}) \implies \mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}}) \subseteq \mathcal{J}$. Claim 2 can be shown by using Example 4.13 as a counter-example. In preparation to do so, let s denote the role *superior*, and W, B, R denote the concepts *Worker*, *Boss* and *Responsible* respectively, also let $\mathcal{K} = \mathcal{K}_{ex1}$, $\mathcal{T} = \mathcal{T}_{ex1}$ and $\mathcal{D} = \mathcal{D}_{ex1}$ for brevity and recall that $\mathcal{D}_W = \mathcal{D}$. It needs to be verified, that

$$\forall \mathcal{J} \in typ^{\mathsf{max}}(\{\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}})\}).\mathcal{J} \models^{rel} W \sqsubseteq\!\!\!\!\sim \exists s.R$$

It is not hard to see that this claim holds if $(d^{\mathcal{D}}_W, d^{\mathcal{D}_B}_B) \in s^{\mathcal{J}}$ for all $\mathcal{J} \in typ^{\mathsf{max}}(\{\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}})\})$.

In order to show that $\forall \mathcal{J} \in typ^{\mathsf{max}}(\{\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}})\}).(d^{\mathcal{D}}_W, d^{\mathcal{D}_B}_B) \in s^{\mathcal{J}}$ holds, we proceed by contradiction and assume that $\exists \mathcal{I} \in typ^{\mathsf{max}}(\{\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}})\}).(d^{\mathcal{D}}_W, d^{\mathcal{D}_B}_B) \notin s^{\mathcal{I}}$, then the interpretation $\mathcal{I}' = \mathcal{I}[s/s^{\mathcal{I}} \cup \{(d^{\mathcal{D}}_W, d^{\mathcal{D}_B}_B)\}]$ is clearly in $typ(\mathcal{I})$ where $X^{\mathcal{I}} = X^{\mathcal{I}'}$ for every left- and right-hand side X of inclusion statements in $\mathcal{T}$ and $\mathcal{D}$, i.e. $\mathcal{I}' \models \mathcal{K}$, hence $\mathcal{I}' \in mc(\mathcal{I}')$, i.e. $mc(\mathcal{I}') \neq \emptyset$. Therefore Condition 1. for typicality upgrade operators from Definition 4.26 applies to $\mathcal{I}$, contradicting that $\mathcal{I} \in typ^{\mathsf{max}}(\{\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}})\})$. □

The counter-example used in this proof gives evidence to the claim that nested *relevant* subsumption is stronger than its materialisation-based counter part. Whether this is also the case for the inferentially weaker nested *rational* subsumption is to be answered next.

4.3.2. Nested Rational Subsumption

Analogous to the case of nested relevant entailment, we need to characterise nested rational entailment for a DKB $\mathcal{K}$, based on the rational domain $\Delta^{\mathcal{K}}_{\text{rat}}$ from Definition 4.17 and the set of maximal typicality models obtained from the minimal typicality model over $\Delta^{\mathcal{K}}_{\text{rat}}$.

Definition 4.30 (Defeasible subsumption under nested rational semantics, $\models^{(rat,nest)}$). Let $Mod_{(rat,nest)}(\mathcal{K}) = typ^{\max}(\{\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rat}})\})$. A defeasible knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ entails a defeasible subsumption $C \mathrel{\sqsubset\!\!\!\sim} D$ under nested rational semantics (written $\mathcal{K} \models^{(rat,nest)} C \mathrel{\sqsubset\!\!\!\sim} D$) *iff* $\mathcal{I} \models^{rat} C \mathrel{\sqsubset\!\!\!\sim} D$ for all $\mathcal{I} \in Mod_{(rat,nest)}(\mathcal{K})$.

It remains to show that reasoning with maximal typicality models supports strictly more entailments than reasoning with all typicality models, hence materialisation-based reasoning under rational closure.

Theorem 4.31. *For two $\mathcal{EL}_\bot$ concepts C, D and an $\mathcal{EL}_\bot$ DKB $\mathcal{K}$ the following holds:*

1. $\mathcal{K} \models^{(rat,mat)} C \mathrel{\sqsubset\!\!\!\sim} D \implies \mathcal{K} \models^{(rat,nest)} C \mathrel{\sqsubset\!\!\!\sim} D$, *and*
2. $\mathcal{K} \models^{(rat,mat)} C \mathrel{\sqsubset\!\!\!\sim} D \not\Longleftarrow \mathcal{K} \models^{(rat,nest)} C \mathrel{\sqsubset\!\!\!\sim} D$

Proof. The proof works analogous to the proof of Theorem 4.29. Claim 1 simply follows from Theorem 4.16 and the fact that $\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rat}}) \subseteq \mathcal{J}$ for all $\mathcal{J} \in typ^{\max}(\{\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rat}})\})$. Claim 2 can be shown with $\mathcal{K}_{ex1}$ from Example 3.2, using the same abbreviations as before. Theorem 4.21 and the consequences discussed in Example 3.2 show that $d_B^{\mathcal{D}_1} \in R^{\mathcal{I}(\Delta_{rat}^{\mathcal{K}_{ex1}})}$. Assume for a contradiction, $\mathcal{I} \in typ^{\max}(\{\mathcal{I}(\Delta_{rat}^{\mathcal{K}_{ex1}})\})$ with $(d_W^{\mathcal{D}_0}, d_B^{\mathcal{D}_1}) \notin s^{\mathcal{I}}$. It is easy to see that the interpretation $\mathcal{J} = \mathcal{I}[s/s^{\mathcal{I}} \cup \{(d_W^{\mathcal{D}_0}, d_B^{\mathcal{D}_1})\}]$ is a safe model of $\mathcal{K}_{ex1}$, contradicting the maximality of $\mathcal{I}$. Therefore all maximal typicality models $\mathcal{J} \in typ^{\max}(\{\mathcal{I}(\Delta_{rat}^{\mathcal{K}_{ex1}})\})$ need to satisfy $(d_W^{\mathcal{D}_0}, d_B^{\mathcal{D}_1}) \in s^{\mathcal{J}}$ and therefore,

$$\mathcal{J} \models Worker \mathrel{\sqsubset\!\!\!\sim} \exists superior.Responsible, \text{ i.e. } \mathcal{K}_{ex1} \models^{(rat,nest)} Worker \mathrel{\sqsubset\!\!\!\sim} \exists superior.Responsible.$$

However, as covered before, $\mathcal{K}_{ex1} \not\models^{(rat,mat)} Worker \mathrel{\sqsubset\!\!\!\sim} \exists superior.Responsible$. □

In this subsection we defined defeasible subsumption under new semantics, namely under nested coverage combined with either relevant or rational strength. These semantics are based on maximal typicality models that we have originally introduced in [11, 12] and that make use of the "typicality dimension" of typicality domains to accommodate different subsets of the DBox $\mathcal{D}$. Now having achieved these results for terminological reasoning, a natural question is how the computational complexity of such reasoning is. This is answered in Section 6. Another follow-up question is how to extend this approach to assertional reasoning. This defeasible instance checking is what we address next.

5. Typicality Interpretations for Defeasible Instance Checking

In this section we want employ typicality interpretations to decide defeasible instance checking under the four semantics considered in the last section. We develop algorithms for the four semantics as for defeasible subsumption. Again, we begin with propositional coverage, but, here, first together with rational and then with relevant strength. Then we turn to nested coverage of defeasible information and again combine it first with rational and then with relevant strength. The starting point of our investigation on defeasible instance checking is the materialisation-based approach by Casini et al. in [4] and our variant of it adapted to $\mathcal{EL}_\bot$ in Section 3.2.2.

First we want to "recreate" materialisation-based instance checking by means of typicality interpretations to achieve methods for deciding defeasible instance checking under $(x, prop)$ semantics. Then we apply our mechanism for achieving typicality upgrades on the conceptual information about individuals. For example, for $(\exists r.A)(a)$ we can apply defeasible information to individual a and check whether a is perhaps related via r to a typical instance of A. This kind of consequence from DCIs for role successors is not supported by Casinis approach.

For DLs that allow reasoning based on canonical models, it is a common technique to translate the ABox into an interpretation. In order to do so, we impose (as in Section 3.2.2) the property that ABoxes are conjunction-free, i.e. conjunctions do not appear on the top-level of any concept. This is merely a requirement to ease presentation, but not a restriction, as the conjuncts of a conjunction can be asserted for an individual one by one. So, we assume w.l.o.g. every ABox is conjunction-free. Recall that, given an $\mathcal{EL}_\bot$ DKB $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$, the ABox $\mathcal{A}$ can be translated into an interpretation $\mathcal{I}$ over the domain $\{d_a \mid a \in sig_I(\mathcal{A})\} \cup \{d_E \mid E \in Qc(\mathcal{K})\}$ as follows:

- $A(a) \in \mathcal{A} \implies d_a \in A^{\mathcal{I}}$,
- $(\exists r.E)(a) \in \mathcal{A} \implies (d_a, d_E) \in r^{\mathcal{I}}$ and
- $r(a, b) \in \mathcal{A} \implies (d_a, d_b) \in r^{\mathcal{I}}$.

In general this particular translation does not yield a model of the whole knowledge base, as the concept representative elements d_E do not necessarily belong to $E^{\mathcal{I}}$. However, due to the construction of a minimal typicality model (and Proposition 4.8), it holds for concept representatives, that $d_E^{\mathcal{U}} \in E^{\mathcal{I}(\Delta^{\mathcal{K}})}$ for any $\mathcal{U} \subseteq \mathcal{D}$ (and $d_E^{\mathcal{U}} \in \Delta^{\mathcal{K}}$). Our approach to obtain a canonical model for the whole DKB $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ with a non-empty ABox $\mathcal{A}$ is to combine the interpretation obtained by translation from the ABox with a minimal typicality model for $(\emptyset, \mathcal{T}, \mathcal{D})$.

The role edges such as (d_a, d_E) in the classical model leave a degree of freedom in a typicality interpretation for choosing the role-successor element among all representatives of E with differing typicality. Those edges (initially) point to those representatives that make the least assumptions about typicality. We use $(d_a, d_E^{\emptyset})$, in order to characterise another kind of *minimal* typicality model. After that we show that propositional rational instance checks are equivalent to the materialisation-based approach. We can then use the same typicality upgrade technique as before to obtain not only strictly more inferences than are contained in the rational closure, but also to introduce propositional and nested relevant instance checking. The method for instance checking in defeasible DLs under relevant semantics presented here is the first one for this task.

The ABox interpretation $\mathcal{I}_{\mathcal{A},\mathcal{T}}$, is constructed using classical reasoning over $(\mathcal{A}, \mathcal{T})$, much like classical canonical models or minimal typicality models.

Definition 5.1. Let $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ be a DKB. Define the *ABox interpretation* $\mathcal{I}_{\mathcal{A},\mathcal{T}} = (\Delta^{\mathcal{I}_{\mathcal{A},\mathcal{T}}}, \cdot^{\mathcal{I}_{\mathcal{A},\mathcal{T}}})$, where

- $\Delta^{\mathcal{I}_{\mathcal{A},\mathcal{T}}} = \{d_a \mid a \in sig_I(\mathcal{A})\} \cup \{d_E^{\emptyset} \mid E \in Qc(\mathcal{K})\}$,
- $a^{\mathcal{I}_{\mathcal{A},\mathcal{T}}} = d_a$ for $a \in sig_I(\mathcal{A})$,
- $A^{\mathcal{I}_{\mathcal{A},\mathcal{T}}} = \{d_a \mid (\mathcal{A}, \mathcal{T}) \models A(a)\}$, and
- $r^{\mathcal{I}_{\mathcal{A},\mathcal{T}}} = \{(d_a, d_b) \mid r(a, b) \in \mathcal{A}\} \cup \{(d_a, d_E^{\emptyset}) \mid (\mathcal{A}, \mathcal{T}) \models (\exists r.E)(a)\}$.

for all $A \in sig_C(\mathcal{K})$, $r \in sig_R(\mathcal{K})$ and $E \in Qc(\mathcal{K})$.

ABox interpretations for $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ are defined such that they are quasi-disjoint from typicality interpretations over $(\emptyset, \mathcal{T}, \mathcal{D})$. This enables the use of Proposition 2.6. ABox interpretations are clearly typicality interpretations according to Definition 4.1. We distinguish the following kinds of typicality interpretations:

- *typicality interpretations over an empty ABox*, if its domain contains only concept representatives, i.e. as considered in Section 4.
- *ABox interpretations* contain essentially no information about concept representatives, but only individuals.
- *typicality interpretations over the full DKB* (or over a non-empty ABox) are unions of a typicality interpretation over an empty ABox and an ABox interpretation.

For an arbitrary typicality interpretation $\mathcal{J} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{J}})$ over $\mathcal{K} = (\emptyset, \mathcal{T}, \mathcal{D})$ and the ABox interpretation $\mathcal{I}_{\mathcal{A},\mathcal{T}}$, the common part of both domains is $\Delta^{\mathcal{K}} \cap \Delta^{\mathcal{I}_{\mathcal{A},\mathcal{T}}} = \{d^{\emptyset}_{E} \mid E \in Qc(\mathcal{K})\}$.[10] Since the extensions of concept names in the ABox interpretation do not contain concept representatives and no role edges have a concept representative as the origin (Def. 5.1), quasi-disjointness holds. Therefore, we can consider the union $\mathcal{J} \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}$, for which the property of Proposition 2.6, i.e. $C^{\mathcal{J} \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}} \cap \Delta^{\mathcal{K}} = C^{\mathcal{J}}$, is satisfied. For such a union of a typicality interpretation over an empty ABox and an ABox interpretation, the extension of individual names is clearly taken from the ABox interpretation, i.e. $a^{\mathcal{J} \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}} = a^{\mathcal{I}_{\mathcal{A},\mathcal{T}}}$.

An instance relationship $C(a)$ is satisfied in a typicality interpretation $\mathcal{I}$ just as in the classical case, i.e. $\mathcal{I} \models C(a)$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$. However, to use our notation in a consistent way we say $\mathcal{I} \models C(a)$ *iff* $\mathcal{I} \models^{rat} C(a)$ *iff* $\mathcal{I} \models^{rel} C(a)$. The following lemma shows that our construction is capable of classical reasoning in the same way a *classical* canonical model is, regardless of the specific typicality domain that is being used in the minimal typicality model.

Lemma 5.2. *For a DKB $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$, a minimal typicality model $\mathcal{I}(\Delta^{\mathcal{K}}) = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{I}(\Delta^{\mathcal{K}})})$ over $\mathcal{K}$ and the ABox interpretation $\mathcal{I}_{\mathcal{A},\mathcal{T}} = (\Delta^{\mathcal{I}_{\mathcal{A},\mathcal{T}}}, \cdot^{\mathcal{I}_{\mathcal{A},\mathcal{T}}})$ it holds that*

$$(\mathcal{A}, \mathcal{T}) \models C(a) \textit{ iff } \mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}} \models C(a)$$

for $\mathcal{EL}_{\bot}$ concepts C with $Qc(C) \subseteq Qc(\mathcal{K})$ and $a \in sig_I(\mathcal{A})$.

Proof. We begin by proving the only if direction, w.l.o.g. for $C = A_1 \sqcap \cdots \sqcap A_n \sqcap \exists r_1.E_1 \sqcap \cdots \sqcap \exists r_m.E_m$. For all $1 \leq i \leq n$ and $1 \leq j \leq m$, $(\mathcal{A}, \mathcal{T}) \models C(a)$ implies $(\mathcal{A}, \mathcal{T}) \models A_i(a)$ and $(\mathcal{A}, \mathcal{T}) \models (\exists r_j.E_j)(a)$. It follows by Definition 5.1 that $d_a \in A_i^{\mathcal{I}_{\mathcal{A},\mathcal{T}}}$ and $(d_a, d^{\emptyset}_{E_j}) \in r_j^{\mathcal{I}_{\mathcal{A},\mathcal{T}}}$ Therefore, by the definition of $\cup$ for interpretations, also $d_a \in A_i^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}}$ and $(d_a, d^{\emptyset}_{E_j}) \in r_j^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}}$. By Propositions 4.8 and 2.6 and the fact that $\mathcal{I}_{\mathcal{A},\mathcal{T}}$ is quasi-disjoint from $\mathcal{I}(\Delta^{\mathcal{K}})$, it holds that $d^{\emptyset}_{E_j} \in E_j^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}}$, hence $d_a \in (\exists r_j.E_j)^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}}$ for $1 \leq j \leq m$. Therefore $d_a \in C^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}}$.

We prove the if direction by structural induction on concept C. For the induction start, $C = A$ ($A \in N_C$), $\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}} \models A(a)$ means $d_a \in A^{\mathcal{I}_{\mathcal{A},\mathcal{T}}}$. By Definition 5.1, $(\mathcal{A}, \mathcal{T}) \models A(a)$ must hold. The induction step for conjunction is trivial and for $C = \exists r.E$, assume for some $b \in sig_I(\mathcal{A})$, $d_b \in E^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}}$ *iff* $(\mathcal{A}, \mathcal{T}) \models E(b)$. From $d_a \in (\exists r.E)^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}}$ we know that there is some $e \in \Delta^{\mathcal{K}} \cup \Delta^{\mathcal{I}_{\mathcal{A},\mathcal{T}}}$ for which $(d_a, e) \in r^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}}$ and $e \in E^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}}$. We distinguish two cases for e either being a concept or an individual representative.

Case 1: e represents an individual in $\Delta^{\mathcal{K}} \cup \Delta^{\mathcal{I}_{\mathcal{A},\mathcal{T}}}$. This means in $\Delta^{\mathcal{I}_{\mathcal{A},\mathcal{T}}}$ w.l.o.g. $e = d_b$. By the induction hypothesis, $d_b \in E^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}}$ is equivalent to $(\mathcal{A}, \mathcal{T}) \models E(b)$ and from Definition 5.1, $(d_a, d_b) \in r^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}}$ implies $r(a, b) \in \mathcal{A}$. Combining both observations, it is implied that $(\mathcal{A}, \mathcal{T}) \models (\exists r.E)(a)$.

Case 2: e represents a concept in $\Delta^{\mathcal{K}} \cup \Delta^{\mathcal{I}_{\mathcal{A},\mathcal{T}}}$. By the construction of $\mathcal{I}_{\mathcal{A},\mathcal{T}}$ we know w.l.o.g. that $e = d^{\emptyset}_{F}$. By Proposition 2.6 and the fact that $\mathcal{I}_{\mathcal{A},\mathcal{T}}$ is quasi disjoint from $\mathcal{I}(\Delta^{\mathcal{K}})$, it holds that $d^{\emptyset}_{F} \in E^{\mathcal{I}(\Delta^{\mathcal{K}})}$. Then Propositions 3.4 and 4.8 imply $F \sqsubseteq_{\mathcal{T}} E$. From $(d_a, d^{\emptyset}_{F}) \in r^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}}$ and Def. 5.1 we know $(\mathcal{A}, \mathcal{T}) \models (\exists r.F)(a)$ and with $F \sqsubseteq_{\mathcal{T}} E$, it is clear that $(\mathcal{A}, \mathcal{T}) \models (\exists r.E)(a)$. □

The interpretations obtained from union of the ABox interpretation and the minimal typicality model are canonical for instance relationships. They are indeed models for the classical components of the DKB: the ABox and the TBox.

Proposition 5.3. *For a minimal typicality model $\mathcal{I}(\Delta^{\mathcal{K}}) = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{I}(\Delta^{\mathcal{K}})})$ over $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ and the ABox interpretation $\mathcal{I}_{\mathcal{A},\mathcal{T}} = (\Delta^{\mathcal{I}_{\mathcal{A},\mathcal{T}}}, \cdot^{\mathcal{I}_{\mathcal{A},\mathcal{T}}})$ it holds that*

$$\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}} \models (\mathcal{A}, \mathcal{T}).$$

[10] Recall, that we can assume w.l.o.g. that $Qc(\mathcal{K}) = Qc(\mathcal{T})$—as discussed in Section 2.

Proof. By Proposition 2.6 and Definition 2.5 we know that $C^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}} \cap \Delta^{\mathcal{K}} = C^{\mathcal{I}(\Delta^{\mathcal{K}})}$. Therefore, the known property $d_F^{\mathcal{U}} \in F^{\mathcal{I}(\Delta^{\mathcal{K}})}$ (Proposition 4.8) is equivalent to $d_F^{\mathcal{U}} \in F^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}}$. We prove satisfaction of the ABox and TBox separately, beginning with $\mathcal{A}$.

First, observe that for any $a \in sig_I(\mathcal{A})$, $a^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}} = a^{\mathcal{I}_{\mathcal{A},\mathcal{T}}} = d_a$ since $\mathcal{I}(\Delta^{\mathcal{K}})$ does not consider individuals. Therefore, we need to check the following three conditions for $\mathcal{A}$ to be satisfied by $\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}$:

1. $r(a, b) \in \mathcal{A} \implies (d_a, d_b) \in r^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}}$,
2. $A(a) \in \mathcal{A} \implies d_a \in A^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}}$, and
3. $(\exists r.E)(a) \in \mathcal{A} \implies d_a \in (\exists r.E)^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}}$.

Towards 1, it holds that $r(a, b) \in \mathcal{A}$ implies $(d_a, d_b) \in r^{\mathcal{I}_{\mathcal{A},\mathcal{T}}}$ by Definition 5.1 and due to the union of interpretations, $(d_a, d_b) \in r^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}}$. 2 and 3 follow directly from Lemma 5.2.

By Lemma 4.9 and Proposition 2.6 it is clear that $\mathcal{T}$ is satisfied w.r.t. domain elements in $\Delta^{\mathcal{K}}$, it remains to show for $C \sqsubseteq D \in \mathcal{T}$ that $d_a \in C^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}} \implies d_a \in D^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}}$ holds for $a \in sig_I(\mathcal{A})$. $d_a \in C^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}}$ is equivalent to $(\mathcal{A}, \mathcal{T}) \models C(a)$ by Lemma 5.2 and for $C \sqsubseteq D \in \mathcal{T}$, this implies $(\mathcal{A}, \mathcal{T}) \models D(a)$ which in turn is equivalent to $d_a \in D^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}}$. □

For typicality interpretations over the full DKB obtained by union, this result, together with the canonicity for instance checking (Lemma 5.2) and the canonicity of the minimal typicality model for subsumption (Lemma 4.9), means that this kind of interpretations admit classical reasoning. Furthermore, the canonicity of the model $\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}$ w.r.t. classical instance checking allows us to reach an equivalent entailment relation for defeasible rational instance checking as in [4] on the ABox level.

We proceed in the order for rational and relevant semantics opposite to earlier sections, as [4] only characterises defeasible instance checking under rational closure and, since, to the best of our knowledge, answering defeasible instance queries under relevant semantics has never been considered before this paper, introducing it requires more effort.

5.1. Propositional Rational Instance Checking

In Section 3 we have already shown how to construct an extended ABox (to be precise, an extended knowledge base) in $\mathcal{EL}_{\bot}$, to obtain equivalent classical entailments as when using (an $\mathcal{EL}_{\bot}$) ABox and extending it with material implication assertions (c.f. $\mathcal{ALC}$, [4]). In the first part of the present section, we have shown minimal typicality models in conjunction with quasi-disjoint ABox interpretations to be canonical w.r.t. classical consequences. From Lemmas 3.7 and 5.2 we immediately obtain that our semantic characterisation of propositional rational instance containment is equivalent to the materialisation-based approach. Recall that for a typicality model $\mathcal{J}$ over a full DKB, $\mathcal{J} \models^{rat} C(a)$ *iff* $\mathcal{J} \models C(a)$ *iff* $a^{\mathcal{J}} \in C^{\mathcal{J}}$, just as in the classical case.

Definition 5.4 (Defeasible instance checking under propositional rational semantics). For a DKB $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$, a sequence s over $sig_I(\mathcal{A})$, the rational domain $\Delta^{\mathcal{K}}_{\text{rat}}$, and the default assumption extension $\widehat{\mathcal{A}}^{s}_{rat}$(for $\mathcal{EL}_{\bot}$), *propositional rational instance containment* (written $\mathcal{K}, s \models^{(rat,prop)} C(a)$) is characterised as $\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rat}}) \cup \mathcal{I}_{\widehat{\mathcal{A}}^{s}_{rat}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}} \models^{rat} C(a)$.

The main result is a direct consequence of the aforementioned lemmas.

Theorem 5.5. *Materialisation-based rational instance checking coincides with propositional rational instance checking, i.e. for an $\mathcal{EL}_{\bot}$ DKB $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ (over a complete ABox and unfoldable TBox), and a sequence s over $sig_I(\mathcal{A})$,*

$$\mathcal{K}, s \models^{(rat,mat)} C(a) \text{ iff } \mathcal{K}, s \models^{(rat,prop)} C(a)$$

□

5.2. Propositional Relevant Instance Checking

This is new territory, as it has never been discussed in the literature. The semantics for $\mathcal{K}, s \models^{(rel,prop)} C(a)$ are obtained analogous to those for $(rat, prop)$, by swapping the underlying domain of the minimal typicality model (over an empty ABox), and using $\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}})$. It remains to devise an ABox extension that is appropriate to obtain relevant consequences.

A first idea would be to adapt the direct construction of sets $\mathcal{D}_C \subseteq \mathcal{D}$ for $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ and a concept C from Section 3, considering justifications for individuals instead of concepts. Recall that inconsistency of a concept with a subset of the DBox is checked by classical reasoning and materialisation of $\mathcal{D}$, e.g. $\overline{\mathcal{D}} \sqcap C \sqsubseteq_{\mathcal{T}} \bot$. Based on this notion of inconsistency, the set of all C-justifications (minimal subsets of $\mathcal{D}$) is computed and the rank-minimal part (w.r.t. *partition*$(\mathcal{D})$) is removed from each justification to obtain the maximal subset of $\mathcal{D}$, consistent with C, called $\mathcal{D}_C$. In this approach, the two consistent DBox subsets $\mathcal{D}_C, \mathcal{D}_H$ of two different concepts C and H are more or less unrelated, determining one does not rely on the other.[11] The set of all justifications can be determined with so-called axiom pinpointing techniques ($\mathcal{EL}$: [21], $\mathcal{ALC}$: [19]), for subsumption and instance checks alike [18]. We do not enter the excursion into the realm of determining justifications in this article, even though some tailoring to consider defeasible concept inclusions is required but can be discovered with little effort.[12] We define minimal justifications for individuals in the lines of Definition 3.1.

Definition 5.6. Let $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ be a DKB, $\mathcal{J} \subseteq \mathcal{D}$, and $a \in sig_I(\mathcal{A})$. $\mathcal{J}$ is an a-justification w.r.t. $\mathcal{K}$, *iff* $(\mathcal{A} \cup \{D_{E \underset{\sim}{\sqsubset} F}(a) \mid E \underset{\sim}{\sqsubset} F \in \mathcal{J}\}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}})$ is inconsistent and $(\mathcal{A} \cup \{D_{E \underset{\sim}{\sqsubset} F}(a) \mid E \underset{\sim}{\sqsubset} F \in \mathcal{J}'\}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}})$ is not inconsistent for all $\mathcal{J}' \subsetneq \mathcal{J}$.

We extend the definition of *justifications*() to allow for $a \in sig_I(\mathcal{K})$ and return *justifications*$(\mathcal{K}, a) = (\mathcal{J}_1, \ldots, \mathcal{J}_m)$, all a-justifications w.r.t. $\mathcal{K}$. The maximal subset of the DBox $\mathcal{D}$ in $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ that is consistent with an individual a, is then denoted as $\mathcal{D}_{\mathcal{K},a} = \mathcal{D} \setminus \bigcup_{i=1}^{m} min_{r_{\mathcal{K}}}(\mathcal{J}_i)$. The reason why we carry $\mathcal{K}$ into the notation for consistent subsets of $\mathcal{D}$ when considering individuals, comes from the issue with explicit edges between individuals $r(a, b) \in \mathcal{A}$, discussed and illustrated in Section 3. The consistent subset of $\mathcal{D}$ for an individual a may be different after having "enriched" another individual b. This is precisely the reason why the naive idea to simply enrich every individual in the given sequence with its relevance based consistent subset of $\mathcal{D}$ is not compatible with the goal to extend rational semantics. Consider for example $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ with $\mathcal{A} = \{A(a), r(b, a)\}$, $\mathcal{T} = \{A \sqcap X \sqsubseteq \bot, B \sqcap \exists r.B \sqsubseteq \bot\}$ and $\mathcal{D} = \{\top \underset{\sim}{\sqsubset} B, \top \underset{\sim}{\sqsubset} X\}$ and the sequence of individuals (a, b). When determining the consistent subset of a w.r.t. $\mathcal{K}$ as it is initially given, the minimal relevant approach would determine $\mathcal{D}_{\mathcal{K},a} = \{\top \underset{\sim}{\sqsubset} B\}$, as $\top \underset{\sim}{\sqsubset} X$ is the only DCI causing an inconsistency with a *at that point*. For the same reason, rational semantics would not allow any DCIs to be satisfied for a, since both DCIs are of the same exceptionality rank (*partition*$(\mathcal{D}) = (\mathcal{D})$). When considering enrichments for the individual b, the DCIs that a is (actively) satisfying have to be taken into account. At this point, $B(a)$ holds for relevant semantics, but not for rational semantics. Therefore, $\top \underset{\sim}{\sqsubset} B$ would cause an inconsistency for b in the current state of ABox extension in relevant semantics. As a result, $B(b)$ can be obtained from rational semantics but not from relevant. This is problematic as the goal of relevant reasoning is to extend rational reasoning (i.e. extend obtained consequences) while "removing" the effect of inheritance blocking. Without at least obtaining rational consequences from relevant semantics, the two become nearly incomparable and thus difficult to put in perspective. For this reason, we propose that the relevant ABox extension needs to build on top of the rational ABox extension, in order to guarantee strictly stronger entailment. Overall, the ABox extension can then be seen as a two-step enrichment. First, defeasible information is added in a rough, DBox-ranked-partition style (including inheritance blocking), and in a second pass over the sequence of individuals, consistent DCIs are determined in a more fine-grained way, in order to combat inheritance blocking.

[11]Note, that for $C \sqsubseteq_{\mathcal{T}} H$ it is could be that $\mathcal{D}_C \subsetneq \mathcal{D}_H$ (not obvious), however since the order in which the DBox subsets are computed is irrelevant, they are deemed unrelated w.r.t. their construction.

[12]Finding justifications for $\mathcal{K} \models \overline{\mathcal{D}} \sqcap C \sqsubseteq \bot$ produces subsets of $\mathcal{K}$ not of $\mathcal{D}$, with our TBox extension however, testing $\mathcal{T}_{\mathcal{D}}(C) \models C_{\mathcal{D}} \sqsubseteq \bot$, one could isolate the part of the justifications containing GCIs $C_{\mathcal{D}} \sqcap E \sqsubseteq F$ to obtain a subset of $\mathcal{D}$.

Algorithm 2 makes clear how the ABox extension is computed based on relevance w.r.t. a given sequence over the individuals in $sig_I(\mathcal{A})$. Recall that $\widehat{\mathcal{A}}^s_{rat}$ is the extended $\mathcal{EL}_\bot$ ABox returned by Algorithm 1.

Algorithm 2: Computation of default assumption extension $\widehat{\mathcal{A}}^s_{rel}$ (relevant)

Input: Sequence s on $sig_I(\mathcal{A})$, w.l.o.g. $s = (a_1, \dots, a_n)$, $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$
Output: Default assumption extension $\widehat{\mathcal{A}}^s_{rel}$

```
A_0 := Â^s_rat
for a_i ∈ (a_1, ..., a_n) do
  | A_i := A_{i-1} ∪ {D_{G⊑~H}(a_i) | G ⊑~ H ∈ D_{(A_{i-1},T,D),a}}
end
return A_n
```

$\mathcal{A}_0 := \widehat{\mathcal{A}}^s_{rat}$
for $a_i \in (a_1, \dots, a_n)$ **do**
$\quad \mathcal{A}_i := \mathcal{A}_{i-1} \cup \{D_{G \precsim H}(a_i) \mid G \precsim H \in \mathcal{D}_{(\mathcal{A}_{i-1},\mathcal{T},\mathcal{D}),a}\}$
end
return $\mathcal{A}_n$

In the following, we denote the returned ABox from Algorithm 2 as $\widehat{\mathcal{A}}^s_{rel}$ and we show superiority (in terms of number of consequences obtained) of relevant over rational semantics.

Definition 5.7 (Defeasible instance checking under propositional relevant semantics). For an $\mathcal{EL}_\bot$ DKB $\mathcal{K} = (\mathcal{A}, \mathcal{T}, D)$, a concept C, an individual $a \in sig_I(\mathcal{A})$ and the extended ABox $\widehat{\mathcal{A}}^s_{rel}$ obtained from $\mathcal{K}$ and s using Algorithm 2, propositional relevant instance query entailment is characterised as follows:

$$\mathcal{K}, s \models^{(rel,prop)} C(a) \textit{ iff } \mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}}) \cup \mathcal{I}_{\widehat{\mathcal{A}}^s_{rel}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}} \models C(a).$$

We show the following example to illustrate how the default assumption extension $\widehat{\mathcal{A}}^s_{rel}$, computed based on relevant semantics, does not exhibit inheritance blocking as the rational extension of the ABox does.

Example 5.8 (Minimal Typicality Model with non-empty ABox).
Consider the DKB $\mathcal{K}_{ex3} = \{\mathcal{A}, \mathcal{T}_{ex1}, \mathcal{D}_{ex1}\}$, extending $\mathcal{K}_{ex1}$ by the ABox $\mathcal{A} = \{Worker(bob), Boss(alice)\}$. We use obvious abbreviations of concept and role names once more to improve readability. Clearly, as bob and alice are not related, any default assumption extension of $\mathcal{A}$ will be independent of the sequence of individuals s. The default assumption extension $\widehat{\mathcal{A}}^s_{rat}$ extends $\mathcal{A}$ by the assertions $D_{W \precsim \exists s.B}(bob)$, $D_{W \precsim P}(bob)$ and $D_{B \precsim R}(bob)$ for the individual bob. For alice, only $D_{B \precsim R}(alice)$ is added, because the concept assertion $D_{W \precsim \exists s.B}(alice)$ leads to the knowledge base $(\mathcal{A} \cup \{D_{W \precsim \exists s.B}(alice)\}, \mathcal{T}_{ex1} \cup \mathcal{T}_{\mathcal{D}_{ex1}})$ to be inconsistent. The rough granularity of the considered DBox subsets in rational semantics then also leads to the exclusion of the assertion $D_{W \precsim P}(alice)$ in $\widehat{\mathcal{A}}^s_{rat}$. $\widehat{\mathcal{A}}^s_{rel}$ then extends (by Algorithm 2) $\widehat{\mathcal{A}}^s_{rat}$ in two iterations. Processing the individual bob may not lead to further extensions, as bob already has all the corresponding DCI assertions from $\mathcal{D}_{ex1}$. Processing alice however, extends $\widehat{\mathcal{A}}^s_{rat}$ by the assertion $D_{W \precsim P}(alice)$, because no (minimal) alice-justification w.r.t. $(\widehat{\mathcal{A}}^s_{rat}, \mathcal{T}_{ex1} \cup \mathcal{T}_{\mathcal{D}_{ex1}})$ contains $D_{W \precsim P}(alice)$.

Since $Boss \sqsubseteq Worker$, $Boss(alice)$ and $D_{W \precsim P}(alice)$ are then contained in the "relevant extension" of $\mathcal{K}_{ex1}$ but not in the "rational extension", we can conclude

$$\mathcal{I}(\Delta^{\mathcal{K}_{ex1}}_{rel}) \cup \mathcal{I}_{\widehat{\mathcal{A}}^s_{rel}, \mathcal{T}_{ex1} \cup \mathcal{T}_{\mathcal{D}_{ex1}}} \models Productive(alice)$$

however,

$$\mathcal{I}(\Delta^{\mathcal{K}_{ex1}}_{rat}) \cup \mathcal{I}_{\widehat{\mathcal{A}}^s_{rat}, \mathcal{T}_{ex1} \cup \mathcal{T}_{\mathcal{D}_{ex1}}} \not\models Productive(alice).$$

Figure 3 shows the minimal typicality model $\mathcal{I}(\Delta^{\mathcal{K}_{ex1}}_{rel})$ in union with the ABox interpretation $\mathcal{I}_{\widehat{\mathcal{A}}^s_{rel}, \mathcal{T}_{ex1} \cup \mathcal{T}_{\mathcal{D}_{ex1}}}$. The granularity of the default assumption extension for the ABox allows to illustrate which concept assertions have been added in the extension by (visually) associating individual domain elements (d_{alice}, d_{bob}) with one of the DBox subsets. For the rational domain (c.f. Fig. 2), one can imagine an analogous visualisation.

Theorem 5.9. *Propositional relevant instance checking is strictly stronger than propositional rational instance checking i.e.*

$$\mathcal{K}, s \models^{(rat,prop)} C(a) \implies \mathcal{K}, s \models^{(rel,prop)} C(a),$$

the converse does not always hold.

Proof. After applying the definitions of both semantics, we need to show that $\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rat}}) + \mathcal{I}_{\widehat{\mathcal{A}}^s_{rat}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}} \models C(a)$ implies $\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}}) + \mathcal{I}_{\widehat{\mathcal{A}}^s_{rel}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}} \models C(a)$. By Lemma 5.2 and monotonicity of classical semantics, this implication holds if $\widehat{\mathcal{A}}^s_{rat} \subseteq \widehat{\mathcal{A}}^s_{rel}$. It is easy to see in Algorithm 2 that $\mathcal{A}_i \subseteq \mathcal{A}_j$ for all $1 \leq i < j \leq n$, hence $\widehat{\mathcal{A}}^s_{rat} = \mathcal{A}_0 \subseteq \mathcal{A}_n = \widehat{\mathcal{A}}^s_{rel}$.

Example 5.8 provides a counterexample for the converse. □

This is a strong result, that aligns with the superiority of relevant semantics over rational semantics for defeasible subsumption. A default assumption extension of the ABox based on minimal justifications has been named as an open problem in [8] but has not been introduced in the literature since. We are positive, that the default assumption extension that we introduce here can be adapted to the materialisation-based approach in order to lift propositional-style semantics to the more expressive DL $\mathcal{ALC}$. As for the semantics of subsumption, eradicating inheritance blocking alone does not mitigate the issue of neglecting quantified concepts in propositional style (materialisation-based) semantics. We investigate typicality upgrades of minimal typicality models over non-empty ABoxes next.

5.3. Nested Instance Checking

The investigation towards nested instance checking semantics can afford to be rather small, due the general definitions regarding the upgrade procedure in Section 4, while presenting a major advancement in defeasible KLM-style reasoning. Not only are relevant semantics for instance checking introduced for the first time, but both rational and relevant semantics are properly lifted to the Description Logic $\mathcal{EL}_\bot$ with the nested coverage. The upgrade procedure as described in Section 4 applies to the typicality interpretations $\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}}) \cup \mathcal{I}_{\widehat{\mathcal{A}}^s_{rel}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}}$ and $\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rat}}) \cup \mathcal{I}_{\widehat{\mathcal{A}}^s_{rat}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}}$ just the same. Only a slight adjustment is required. Individual representatives are not syntactically associated with a DBox, hence model completions w.r.t. $\mathcal{K} = (\widehat{\mathcal{A}}^s_{rel}, \mathcal{T}, \mathcal{D})$ do not force individual representatives to keep satisfying DCIs after a typicality upgrade. At the same time, assertions such as $D_{E \sqsubseteq\!\!\!\sim F}(a)$ hold no meaning in the extended ABox without considering $\mathcal{T}_{\mathcal{D}}$. Therefore, for the input DKB $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$, the typicality upgrade procedure w.r.t. e.g. rational semantics always needs to consider $(\widehat{\mathcal{A}}^s_{rat}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}, \mathcal{D})$ as the input. Now, everything required to characterise both types of nested semantics and show their superiority over the propositional-style semantics, exists.

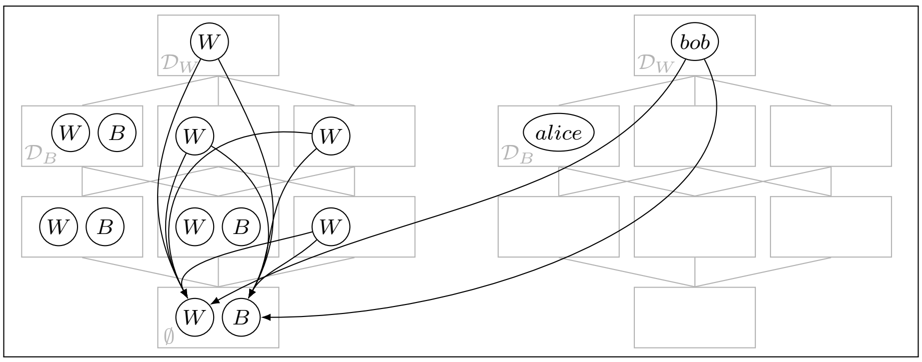

Figure 3: The minimal typicality model $\mathcal{I}(\Delta^{\mathcal{K}_{ex1}}_{rel}) \cup \mathcal{I}_{\widehat{\mathcal{A}}^s_{rel}, \mathcal{T}_{ex1} \cup \mathcal{T}_{\mathcal{D}_{ex1}}}$.

5.3.1. Nested Rational Instance Checking

It only remains to define defeasible instance checking under nested rational semantics and show that the resulting entailments properly extend those obtained by propositional style semantics.

Definition 5.10 (Defeasible instance checking under nested rational semantics). For a given $\mathcal{EL}_\perp$ DKB $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$, a concept C, an individual $a \in sig_I(\mathcal{A})$ and a sequence s over the individuals in $sig_I(\mathcal{A})$, nested rational instance checking is defined as follows. Let $Mod_{(rat,nest)} = typ^{\mathsf{max}}(\{\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rat}}) \cup \mathcal{I}_{\widehat{\mathcal{A}}^s_{rat}, \mathcal{T} \cup \mathcal{T}_\mathcal{D}}\})$.

$$\mathcal{K}, s \models^{(rat,nest)} C(a) \textit{ iff } \forall \mathcal{J} \in Mod_{(rat,nest)}.\mathcal{J} \models^{rat} C(a)$$

We characterise superiority of these semantics over propositional-style semantics through implication of entailment as follows.

Theorem 5.11. *Nested rational instance checking is strictly stronger than propositional rational instance checking, i.e.*

$$\mathcal{K}, s \models^{(rat,prop)} C(a) \implies \mathcal{K}, s \models^{(rat,nest)} C(a),$$

the converse does not hold in general.

Proof. For all interpretations $\mathcal{J} \in typ^{\mathsf{max}}(\{\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rat}}) \cup \mathcal{I}_{\widehat{\mathcal{A}}^s_{rat}, \mathcal{T} \cup \mathcal{T}_\mathcal{D}}\})$, it is clear that $\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rat}}) \cup \mathcal{I}_{\widehat{\mathcal{A}}^s_{rat}, \mathcal{T} \cup \mathcal{T}_\mathcal{D}} \subseteq \mathcal{J}$, since $typ()$ and $mmc()$ only ever *extend* the minimal typicality model over the full DKB as well as $\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rat}}) \cup \mathcal{I}_{\widehat{\mathcal{A}}^s_{rat}, \mathcal{T} \cup \mathcal{T}_\mathcal{D}}}$ and $a^{\mathcal{J}} = a^{\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rat}}) \cup \mathcal{I}_{\widehat{\mathcal{A}}^s_{rat}, \mathcal{T} \cup \mathcal{T}_\mathcal{D}}}$. By Proposition 4.4, $\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rat}}) \cup \mathcal{I}_{\widehat{\mathcal{A}}^s_{rat}, \mathcal{T} \cup \mathcal{T}_\mathcal{D}} \models C(a)$ then clearly implies $\mathcal{J} \models C(a)$. To disprove the converse, consider Example 5.8, with the obvious concept and role name abbreviations. It can be readily seen, that since $\mathcal{A}$ does not contain any role-assertions, the domain element d_{bob} "behaves" the same as $d^{\mathcal{D}}_W$ w.r.t. model completions and typicality extensions. Therefore, an analogous argument as in the proof of Theorem 4.31 shows that $(d_{bob}, d^{\mathcal{D}_2}_B) \in s^{\mathcal{J}}$ must be true in all maximal typicality models $\mathcal{J}$ of $\mathcal{I}(\Delta^{\mathcal{K}_{ex1}}_{rel}) \cup \mathcal{I}_{\widehat{\mathcal{A}}^s_{rel}, \mathcal{T}_{ex1} \cup \mathcal{T}_{\mathcal{D}_{ex1}}}$, hence $\mathcal{K}_{ex3}, s \models (\exists \mathit{superior}.\mathit{Responsible})(\mathit{bob})$ holds for all sequences s over $sig_I(\mathcal{A})$, which cannot be derived by propositional rational semantics. □

5.3.2. Nested Relevant Instance Checking

It only remains to define defeasible instance checking under nested relevant semantics and show that the resulting entailments properly extend those obtained by propositional style semantics. Analogous to nested rational semantics, we define nested relevant semantics based on the relevant domain and the relevance-based ABox extension $\widehat{\mathcal{A}}^s_{rel}$.

Definition 5.12 (Defeasible instance checking under nested relevant semantics). For a given $\mathcal{EL}_\perp$ DKB $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$, a concept C, an individual $a \in sig_I(\mathcal{A})$ and a sequence s over the individuals in $sig_I(\mathcal{A})$, nested rational instance checking is defined as follows. Let $Mod_{(rel,nest)} = typ^{\mathsf{max}}(\{\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}}) \cup \mathcal{I}_{\widehat{\mathcal{A}}^s_{rel}, \mathcal{T} \cup \mathcal{T}_\mathcal{D}}\})$.

$$\mathcal{K}, s \models^{(rel,nest)} C(a) \textit{ iff } \forall \mathcal{J} \in Mod_{(rel,nest)}.\mathcal{J} \models^{rel} C(a)$$

The following main result naturally aligns with Theorem 5.11.

Theorem 5.13. *Nested relevant instance checking is strictly stronger than propositional relevant instance checking, i.e.*

$$\mathcal{K}, s \models^{(rel,prop)} C(a) \implies \mathcal{K}, s \models^{(rel,nest)} C(a),$$

the converse does not hold in general.

Proof. Proving the implication is analogue to the first part of the proof of Theorem 5.11. To disprove the converse, consider Example 5.8, with the obvious concept and role name abbreviations. With an analogous argumentation as in the proofs of Theorem 4.31 and 5.11 we conclude that $(d_{bob}, d_B^{\mathcal{D}_B}) \in s^{\mathcal{J}}$ must be true in all maximal typicality models $\mathcal{J}$ of $\mathcal{I}(\Delta_{rel}^{\mathcal{K}_{ex1}}) \cup \mathcal{I}_{\widehat{\mathcal{A}}^s_{rel}, \mathcal{T}_{ex1} \cup \mathcal{T}_{\mathcal{D}_{ex1}}}$, hence $\mathcal{K}_{ex3}, s \models (\exists superior.(Responsible \sqcap Productive))(bob)$ holds for all sequences s over $sig_I(\mathcal{A})$, which cannot be derived by propositional relevant semantics. □

The final remark in the proof of Theorem 5.13 also shows how defeasible conclusions derived for quantified concepts are generally not subject to inheritance blocking under nested relevant entailments.

6. Complexity of Nested Defeasible Entailments

In this section we are investigating the computational complexity for deciding the entailments under nested rational and relevant semantics. Since propositional semantics coincide with the materialisation-based approach, our results from Section 3.2 prove the claim in [4], that the complexity of materialisation-based (and propositional) reasoning resides with the underlying DL. It is thus polynomial for $\mathcal{EL}_\bot$ w.r.t. rational semantics. The complexity of materialisation-based and propositional relevant semantics appears to be dominated by the computation of all justifications (worst-case exponentially many in the input). The discussion is cut short in [8], since for $\mathcal{ALC}$ this observation might be enough, as classical reasoning in $\mathcal{ALC}$ is EXP–Time complete [22]. For $\mathcal{EL}_\bot$ it might be worth investigating whether a consistent subset of the DBox can be computed without enumerating all justifications, but simply determining whether a single DCI belongs to some justification for every DCI. The latter is shown to be NP–compl. in [18]. Since this investigation would lean too far into the subject of axiom pinpointing, we leave this open for future work.

We shall investigate upper bounds for defeasible subsumption and instance checking under nested rational and relevant semantics by describing algorithms to compute minimal model completions and maximal typicality models, based on a non-deterministic guess of typicality upgrades. In the second part of this section, we present a reduction from the NP–compl. (1-in-3)-positive 3SAT satisfiability problem to non-entailment of defeasible subsumption under nested rational semantics. This lower bound also translates to defeasible instance checking under nested rational semantics with a simple argument. Unfortunately, this lower bound does not immediately translate to the stronger relevant semantics. Finding an appropriate bound on defeasible subsumption and instance checking under nested relevant semantics is highly non-trivial and therefore left as an open question.

As a foundation for our investigation of the computational complexity, we introduce the notion of a *size of a DKB* and of its components. Let $\mathcal{K}$ be a DKB, $\mathcal{A}$ an ABox, $\mathcal{T}$ a TBox, $\mathcal{D}$ a DBox and let C and D be concepts. The size of these is:

- $\|A\| = 1$ for $A \in N_C$,
- $\|\top\| = \|\bot\| = 1$
- $\|C \sqcap D\| = \|C\| + \|D\| + 1$,
- $\|\exists r.C\| = \|C\| + 1$,
- $\|C \sqsubseteq D\| = \|C \sqsubsetsim D\| = \|C\| + \|D\|$,
- $\|C(a)\| = \|C\| + 1$,
- $\|r(a,b)\| = 3$,
- $\|\mathcal{A}\| = \sum_{C(a)\in\mathcal{A}} \|C(a)\| + \sum_{r(a,b)\in\mathcal{A}} \|r(a,b)\|$,
- $\|\mathcal{T}\| = \sum_{C\sqsubseteq D\in\mathcal{T}} \|C \sqsubseteq D\|$,
- $\|\mathcal{D}\| = \sum_{C\sqsubsetsim D\in\mathcal{D}} \|C \sqsubsetsim D\|$, and

- $\|\mathcal{K}\| = \|\mathcal{A}\| + \|\mathcal{T}\| + \|\mathcal{D}\|$.

In the course of this section, we use both, the size of the encoding $\|\cdot\|$ and the standard set cardinality $|\cdot|$ to be precise. Note that for the input DKB $\mathcal{K}$, the size of $\mathcal{K}$ ($\|\mathcal{K}\|$) is polynomial in the cardinality of $\mathcal{K}$ ($|\mathcal{K}|$), hence whenever some result is polynomial (exponential) in $|\mathcal{K}|$, it will be in polynomial (exponential) also in $\|\mathcal{K}\|$.

6.1. Upper Bounds for Nested Rational and Relevant Semantics

As by the previous strategy throughout this article, we introduce general algorithms to do the following:

1. Compute the minimal model completion of some typicality interpretation (be it based on a chain or lattice domain), and
2. Compute a maximal typicality model, starting from some minimal typicality model (be it $\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rat}})$ or $\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}})$).

We can show termination, complexity and correctness of both procedures depending on the size of the underlying typicality domain. Finally, it only remains to determine the size of $\Delta^{\mathcal{K}}_{\text{rat}}$ and $\Delta^{\mathcal{K}}_{\text{rel}}$ and instantiate the general results from the main part of this section.

We present Algorithm 3 to decide the existence of the minimal model completion of a given typicality interpretation $\mathcal{I} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{I}})$ over DKB $\mathcal{K}$, that satisfies Condition 1 of standard typicality interpretations (from Definition 4.3). The algorithm is constructive; it computes the minimal model completion if it exists.

Algorithm 3 takes as an input an interpretation satisfying Condition 1 of Def. 4.3 and a DKB $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ and has the goal to extend $\mathcal{I}$, such that all GCIs (for all $d \in \Delta^{\mathcal{K}}$) and all DCIs (all $\mathcal{U} \subseteq \mathcal{D}$ for every $d^{\mathcal{U}}_F \in \Delta^{\mathcal{K}}$) are satisfied for the appropriate domain elements, in a completion-like fashion. It utilises the functions `Unresolved`$_C$`()` and `Unresolved`$_I$`()` to determine a kind of ToDo list of GCIs and DCIs that are *currently* violated. One iteration of the main while loop (Line 9) selects one domain element with an unresolved GCI (DCI) and extends "the current" interpretation to then satisfy this GCI (DCI). Note, that the algorithm does not consider ABox assertions. Recalling the approach to decide defeasible instance queries under nested semantics, you can see that the construction of an ABox-interpretation uses classical reasoning to eventually satisfy the ABox assertions (i.e. $\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}} \models \mathcal{A}$). During the typicality upgrade procedure, only extensions of the minimal typicality model are considered, hence no assertions in $\mathcal{A}$ can be violated during the upgrade phase.

Proposition 6.1. *Let $\mathcal{I} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{I}})$ be a typicality interpretation (over DKB $\mathcal{K}$) that satisfies Condition 1 of Definition 4.3. Algorithm 3 (Minimal Model Completion) terminates in polynomial time in $|\Delta^{\mathcal{K}}|$ on the inputs $\mathcal{I}, \mathcal{K}$.*

Proof. It is clear that (i) at any given point in Algorithm 3, $\Theta \subseteq \Delta^{\mathcal{K}} \times (\mathcal{T} \cup \mathcal{D})$, which has $|\Delta^{\mathcal{K}}| * (|\mathcal{T}| + |\mathcal{D}|)$ as an upper bound, i.e. it is polynomial in the size of $\Delta^{\mathcal{K}}$. Lines 16–18 are well-defined, as $E_j, E \in Qc(\mathcal{K})$ for $1 \leq j \leq l$, hence $d^{\emptyset}_{E_j}$, $d^{\emptyset}_{E}$ exist by the definition of a typicality domain. Note, that in the following, the domain element d, can be either a concept ($d^{\mathcal{U}}_F$) or individual (d_a) representative. For $(d, G \bowtie H) \in \Theta$ at iteration x, for $H = A_1 \sqcap \cdots \sqcap A_k \sqcap \exists r_1.E_1 \sqcap \cdots \sqcap \exists r_l.E_l$, it holds that $d \in A_i^{\mathcal{I}_{x+1}}$ for all $1 \leq i \leq k$, by the construction of σ_1 (Line 15). Because the input $\mathcal{I}$ satisfies Condition 1 of Definition 4.3 ($d^{\mathcal{U}}_F \in F^{\mathcal{I}}$) and it holds that $\mathcal{I}_x \subseteq \mathcal{I}_{x+1}$ ($x \geq 1$), we know that $d^{\emptyset}_{E_j} \in E_j^{\mathcal{I}_{x+1}}$ (Proposition 4.4). Due to the definition of σ_2 (Line 16) it then follows that $d \in (\exists r_j.E_j)^{\mathcal{I}_{x+1}}$ for all $1 \leq j \leq l$ and thus $d \in H^{\mathcal{I}_{x+1}}$. Hence, (ii) each iteration of the while loop resolves one unsatisfied GCI or DCI for one domain element in $\Delta^{\mathcal{K}}$. That means, once the selected pair $(d, G \bowtie H)$ has been treated in the while loop (say at iteration x), d always belongs to both $G^{\mathcal{I}_y}$ and $H^{\mathcal{I}_y}$ ($y > x$) because the substitutions defined in Lines 15–17 only extend $\mathcal{I}_x$. Combining observations (i) and (ii), we can see that the while loop can be entered at most $|\Delta^{\mathcal{K}}| * (|\mathcal{T}| + |\mathcal{D}|)$ times. The sets returned by `Unresolved`$_C$`()` and `Unresolved`$_I$`()` can each be computed in polynomial time in the size of both inputs (i.e. $|\Delta^{\mathcal{K}}|$ and $\|\mathcal{K}\|$). Therefore, Algorithm 3 terminates in any case after polynomial time in $|\Delta^{\mathcal{K}}| + \|\mathcal{K}\|$. □

Algorithm 3: Minimal Model Completion

Function $\texttt{Unresolved}_C(\mathcal{J}, \mathcal{K})$
  Input: $\mathcal{J} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{J}})$: a typicality interpretation, $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$: a DKB
  return $\{(d_F^{\mathcal{U}}, G \bowtie H) \in \Delta^{\mathcal{K}} \times (\mathcal{T} \cup \mathcal{U}) \mid d_F^{\mathcal{U}} \in G^{\mathcal{J}} \wedge d_F^{\mathcal{U}} \notin H^{\mathcal{J}}\}$
Function $\texttt{Unresolved}_I(\mathcal{J}, \mathcal{K})$
  Input: $\mathcal{J} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{J}})$: a typicality interpretation, $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$: a DKB
  return $\{(d_a, G \sqsubseteq H) \in \Delta^{\mathcal{K}} \times \mathcal{T} \mid d_a \in G^{\mathcal{J}} \wedge d_a \notin H^{\mathcal{J}}\}$
Algorithm *Minimal Model Completion*
  Input: $\mathcal{I} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{I}})$: a typicality interpretation satisfying Condition 1 of Definition 4.3,
    $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$: a DKB
  Output: $mmc(\mathcal{I})$
  $\Theta := \texttt{Unresolved}_C(\mathcal{I}, \mathcal{K}) \cup \texttt{Unresolved}_I(\mathcal{I}, \mathcal{K})$
  $\mathcal{I}_1 := \mathcal{I}$
  $x := 1$
  while $\Theta \neq \emptyset$ **do**
    if $\exists(d, G \sqsubseteq \bot) \in \Theta$ **then**
      return `false`
    end
    Select $(d, G \bowtie H)$ from Θ.
    W.l.o.g. $H = A_1 \sqcap \cdots \sqcap A_k \sqcap \exists r_1.E_1 \sqcap \cdots \sqcap \exists r_l.E_l$ for $A_i \in sig_C(\mathcal{K})$ and
      $E_j \in Qc(\mathcal{K})$ $(1 \leq i \leq k,\ 1 \leq j \leq l)$.
    $\sigma_1 := \{A_i / A_i^{\mathcal{I}_x} \cup \{d\} \mid 1 \leq i \leq k\}$
    $\sigma_2 := \{r_j / r_j^{\mathcal{I}_x} \cup \{(d, d_{E_j}^{\emptyset})\} \mid 1 \leq j \leq l\}$
    $\sigma_3 := \{r / r^{\mathcal{I}_x[\sigma_1 \cup \sigma_2]} \cup \{(d, d_E^{\emptyset}) \mid d \in (\exists r.E)^{\mathcal{I}_x[\sigma_1 \cup \sigma_2]}, E \in Qc(\mathcal{K})\} \mid r \in sig_R(\mathcal{K})\}$
    $\mathcal{I}_{x+1} := \mathcal{I}_x[\sigma_1 \cup \sigma_2][\sigma_3]$
    $\Theta := \texttt{Unresolved}_C(\mathcal{I}_{x+1}, \mathcal{K}) \cup \texttt{Unresolved}_I(\mathcal{I}_{x+1}, \mathcal{K})$
    $x := x + 1$
  end
  $\mathcal{I}_{fin} := \mathcal{I}_x$
  return $\mathcal{I}_{fin}$

Next we show soundness and completeness of Algorithm 3. More precisely, we show that Algorithm 3 either computes the minimal model completion or returns false, if no model completion exists. For an interpretation $\mathcal{J}$ to be a model completion of the input $\mathcal{I}$ to Algorithm 3, it has to satisfy in particular $\mathcal{A}$, for the input $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$. As described before, this algorithm will only be used during the typicality upgrade phase, hence eventually, the input $\mathcal{I}$ will always be an extension of a minimal model completion (over the full DKB). That such a minimal typicality model satisfies $\mathcal{A}$ follows directly from Lemma 5.2, therefore any extension of a minimal typicality model satisfies $\mathcal{A}$ as well (Prop. 4.4). Of course the complexity of computing the minimal typicality model as well as an ABox model will be discussed in the specific sections for rational and relevant semantics, here we focus on general results for the typicality upgrade procedure.

Proposition 6.2. *For a typicality interpretation $\mathcal{I} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{I}})$ over the DKB $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ satisfying Condition 1 of Definition 4.3 and $\mathcal{I} \models \mathcal{A}$, either $mc(\mathcal{I}) = \emptyset$ or $\mathcal{I}_{fin}$ as returned by Algorithm 3 is the minimal model completion of $\mathcal{I}$.*

Proof. Since $\mathcal{I} \models \mathcal{A}$, Proposition 4.4 and $\mathcal{I} \subseteq \mathcal{I}_x$ $(x \geq 1)$ imply $\mathcal{I}_{fin} \models \mathcal{A}$, hence, in order to show that $\mathcal{I}_{fin} \models \mathcal{K}$, we need to show that $\mathcal{I}_{fin} \models \mathcal{T}$ and $\mathcal{I}_{fin}, d_F^{\mathcal{U}} \models \mathcal{U}$. For $\mathcal{I}_{fin}$ to be a model completion of the input $\mathcal{I}$, we also need to show that $\mathcal{I}_{fin}$ is standard.

We begin by proving an intermediate claim that will aid us in proving this proposition. We claim that the set of model completions of the input $\mathcal{I}$ and all extensions $\mathcal{I}_x$ $(x \geq 1)$ computed during a run of Algorithm

3 on $\mathcal{I}$ and $\mathcal{K}$, is the same throughout the run. Eventually this allows to conclude that if $\mathcal{I}_{fin}$ is a model completion of itself, it is the minimal model completion w.r.t. its set of model completion and therefore the minimal model completion of the input $\mathcal{I}$.

Claim: $mc(\mathcal{I}_x) = mc(\mathcal{I}_{x+1})$ for $(x \geq 0)$.

The "$\supseteq$" direction follows from the fact that $\mathcal{I}_x \subseteq \mathcal{I}_{x+1}$, which is easy to see from Lines 15 to 18 in Algorithm 3. For "$\subseteq$" we have to show that one application of the while loop on $\mathcal{I}_x$ preserves all model completions for $\mathcal{I}_{x+1}$. For $\mathcal{J} \in mc(\mathcal{I}_x)$ it holds that $\mathcal{I}_x \subseteq \mathcal{J}$ by 1 in Def. 4.23. Because $d \in G^{\mathcal{I}_x}$ holds for all $(d, G \bowtie H) \in \texttt{Unresolved}_C(\mathcal{I}_x) \cup \texttt{Unresolved}_I(\mathcal{I}_x)$ it follows from Claim 1 in Proposition 4.4 that $d \in G^{\mathcal{J}}$. Since $\mathcal{J} \models \mathcal{K}$ (Def. 4.23), in either case, $G \sqsubseteq H \in \mathcal{T}$ and $G \sqsubsetapprox H \in \mathcal{U}$ (if applicable for $d = d_F^{\mathcal{U}}$), d has to satisfy $G \bowtie H$ in $\mathcal{J}$, and since $d \in G^{\mathcal{J}}$ it must hold that $d \in H^{\mathcal{J}}$. We distinguish two cases for the concept H:

1. $H = A_1 \sqcap \cdots \sqcap A_k \sqcap \exists r_1.E_1 \sqcap \cdots \sqcap \exists r_l.E_l$ (this structure can be assumed w.l.o.g. if $H \neq \bot$, which is covered by the second case)
   In this case, $d \in A_i^{\mathcal{J}}$ for all $1 \leq i \leq k$ and $d \in (\exists r_j.E_j)^{\mathcal{J}}$ for all $1 \leq j \leq l$. Hence, by Condition 3 of model completions (Def. 4.23) it holds that $(d, d_{E_j}^{\emptyset}) \in r_j^{\mathcal{J}}$. The element $d_{E_j}^{\emptyset}$ exists in the typicality domain by its definition and the fact that $E_j \in Qc(\mathcal{K})$. The difference between $\mathcal{I}_x$ and $\mathcal{I}_{x+1}$ is only determined by Lines 15–17, which is in the first step, adding d to the extension of all A_i ($1 \leq i \leq k$, Line 15) and adding role successors $(d, d_{E_j}^{\emptyset})$ to the extension of all r_j ($1 \leq j \leq l$, Line 16). At this point, it is shown that $\mathcal{I}_x[\sigma_1 \cup \sigma_2] \subseteq \mathcal{J}$ holds. Therefore, Proposition 4.4 implies $(\exists r.E)^{\mathcal{I}_x[\sigma_1 \cup \sigma_2]} \subseteq (\exists r.E)^{\mathcal{J}}$ for all $r \in sig_R(\mathcal{K})$ and $E \in Qc(\mathcal{K})$. Since $\mathcal{J}$ is standard, all edges $(d, d_E^{\emptyset})$ that are being added to $\mathcal{I}_x[\sigma_1 \cup \sigma_2]$ by σ_3 already exist in $\mathcal{J}$, hence $\mathcal{I}_x[\sigma_1 \cup \sigma_2][\sigma_3] = \mathcal{I}_{x+1} \subseteq \mathcal{J}$, and therefore, $\mathcal{J} \in mc(\mathcal{I}_{x+1})$.
2. $H = \bot$.
   Using the same argument as before, all model completions $\mathcal{J}$ of $\mathcal{I}_x$ have $d \in G^{\mathcal{J}}$ and must therefore satisfy $d \in \bot^{\mathcal{J}}$. Since the latter is not possible, there cannot be any model completions of $\mathcal{I}_x$ and thus extending it in any way, in particular extending it to $\mathcal{I}_{x+1}$ does not allow for new model completions. It follows that $mc(\mathcal{I}_x) = mc(\mathcal{I}_{x+1}) = \emptyset$.

This concludes the proof of the claim.

By allowing x to be any iteration in this claim, it is clear that $mc(\mathcal{I}_1) = mc(\mathcal{I}_2) = \ldots = mc(\mathcal{I}_x)$ for all $x \geq 1$ until termination of Algorithm 3, which is ensured by Proposition 6.1. The claim implies two things: first, when there is an iteration x such that $(d, G \sqsubseteq \bot) \in \texttt{Unresolved}_C(\mathcal{I}_x) \cup \texttt{Unresolved}_I(\mathcal{I}_x)$, then $mc(\mathcal{I}_x) = \emptyset$ and thus $mc(\mathcal{I}) = \emptyset$. And secondly, assume there is no iteration x with $(d, G \sqsubseteq \bot) \in \texttt{Unresolved}_C(\mathcal{I}_x) \cup \texttt{Unresolved}_I(\mathcal{I}_x)$. Upon termination, $\texttt{Unresolved}_C(\mathcal{I}_{fin}) = \texttt{Unresolved}_I(\mathcal{I}_{fin}) = \emptyset$ which implies $\mathcal{I}_{fin} \models \mathcal{T}$ as well as $\mathcal{I}_{fin}, d_F^{\mathcal{U}} \models \mathcal{U}$, hence $\mathcal{I}_{fin} \models \mathcal{K}$, since $\mathcal{I}_{fin} \models \mathcal{A}$ has been established in the beginning.

It remains to show that $\mathcal{I}_{fin}$ satisfies Condition 3 of 4.23. In general, consider any typicality interpretation $\mathcal{J}$ satisfying Condition 1 of Definition 4.3 and $\sigma = \{r/r^{\mathcal{J}} \cup \{(d, d_E^{\emptyset}) \mid d \in (\exists r.E)^{\mathcal{J}}, E \in Qc(\mathcal{K})\} \mid r \in sig_R(\mathcal{K})\}$ (Line 17). It clearly holds that $\mathcal{J}[\sigma]$ is a standard typicality interpretation. As discussed before, $\mathcal{I}_{fin}$ also satisfies Condition 1 of Definition 4.3 and since it has been obtained by applying σ_3 (Line 17) to some previous interpretation, it must be standard.

Finally, it holds that $\mathcal{I}_{fin} \in mc(\mathcal{I}_{fin})$, which implies $\mathcal{I}_{fin} \subseteq \mathcal{J}$ for all $\mathcal{J} \in mc(\mathcal{I}_{fin})$ and thus $\mathcal{I}_{fin} = mmc(\mathcal{I}_{fin})$. Since $mc(\mathcal{I}) = mc(\mathcal{I}_{fin})$ (by our claim), it holds that $\mathcal{I}_{fin} = mmc(\mathcal{I})$. □

The following corollary combines the results from Propositions 4.24 and 6.1.

Corollary 6.3. *For a standard typicality interpretation $\mathcal{I} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{I}})$ over the DKB $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ (satisfying the ABox $\mathcal{A}$), the minimal model completion $mmc(\mathcal{I})$ can be computed in polynomial time in the size of $\Delta^{\mathcal{K}}$, if it exists. Likewise, its existence can be decided in polynomial time.*

To construct the minimal typicality model for both the rational and relevant typicality domain (over the empty ABox), we need to decide the subsumptions $F_{\mathcal{U}} \sqsubseteq_{\mathcal{T}_{\mathcal{U}}(F)} A$ and $F_{\mathcal{U}} \sqsubseteq_{\mathcal{T}_{\mathcal{U}}(F)} \exists r.E$ for all $d_F^{\mathcal{U}} \in \Delta_x^{\mathcal{K}}$ ($x \in \{rat, rel\}$), $A \in sig_C(\mathcal{K})$, $r \in sig_R(\mathcal{K})$ and $E \in Qc(\mathcal{K})$. These are

$$|\Delta_x^{\mathcal{K}}| * (|sig_C(\mathcal{K})| + |sig_R(\mathcal{K})| * |Qc(\mathcal{K})|),$$

i.e. polynomially many classical subsumption checks in $|\Delta_x^{\mathcal{K}}|$ and $\|\mathcal{K}\|$ ($|\mathcal{T}_{\mathcal{U}}(F)|$ is linear in $|\mathcal{K}|$). For $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$, the interpretation $\mathcal{I}_{\mathcal{A},\mathcal{T}}$ has a domain of size $|sig_I(\mathcal{A})| + |Qc(\mathcal{K})|$, i.e. linear in $\|\mathcal{K}\|$. A number of classical instance checks $(\mathcal{A}, \mathcal{T}) \models A(a)$ and $(\mathcal{A}, \mathcal{T}) \models (\exists r.E)(a)$ are required for every element $d_a \in \Delta^{\mathcal{I}_{\mathcal{A},\mathcal{T}}}$. To be specific, the number of classical entailment checks required to construct the ABox interpretation is

$$\big(|sig_C(\mathcal{K})| + \big(|sig_R(\mathcal{K})| * |Qc(\mathcal{K})|\big)\big) * |sig_I(\mathcal{A})|,$$

i.e. polynomially many in $\|\mathcal{K}\|$. When combining a minimal typicality model $\mathcal{I}(\Delta^{\mathcal{K}})$ over an empty ABox and an ABox interpretation for the same knowledge base $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$, the size of the combined domain is dominated by the size of $\Delta^{\mathcal{K}}$,[13] i.e. $\Delta^{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}}$ is only linearly (in $|\mathcal{K}|$) bigger than $\Delta^{\mathcal{K}}$. Recall that, both $\mathcal{I}(\Delta^{\mathcal{K}})$ and $\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}$ are considered minimal typicality models for $\Delta^{\mathcal{K}}$ over $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$.

Proposition 6.4. *The number of maximal typicality models for a given minimal typicality model* $\mathcal{I} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{I}})$ *is at most exponential in* $|\Delta^{\mathcal{K}}|$.

Proof. Every maximal typicality model (obtained with the fixpoint of the T-operator) is a minimal model completion of some typicality interpretation. Secondly, the maximum number of role edges in $\Delta^{\mathcal{K}}$ using only role names from $sig_R(\mathcal{K})$ is $|\Delta^{\mathcal{K}} \times \Delta^{\mathcal{K}}| * |sig_R(\mathcal{K})|$, which is polynomial in $|\Delta^{\mathcal{K}}|$ (and $\|\mathcal{K}\|$). Hence, the upper bound on the number of minimal model completions using distinct role edge sets is therefore exponential in $|\Delta^{\mathcal{K}}|$ (all subsets of all possible role edges) and the size of $typ^{\mathsf{max}}(\{\mathcal{I}\})$ is clearly contained in this upper bound. □

This upper bound on maximal typicality models seems high, but it shall suffice for showing the overall upper bound on the complexity of deciding nested defeasible subsumptions as well as instance checks. For exponentially many maximal typicality models, it takes at least exponential time to compute and iterate over all maximal typicality models and check whether the queried subsumption is entailed by all of them. The alternative is to guess a maximal typicality model of the minimal typicality model $\mathcal{I}$ (be it $\mathcal{I}(\Delta^{\mathcal{K}})$ or $\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}$ if applicable), which is a structure of polynomial size in $|\Delta^{\mathcal{K}}|$ and check whether it does *not* satisfy the query. This provides a procedure for deciding the complement of our initial problem, i.e. non-entailment of a defeasible subsumption query or instance check. Due to the intricate interplay between the "operations" typicality extension and minimal model completion for constructing $typ^{\mathsf{max}}(\{\mathcal{I}\})$ it is difficult to decide whether a guessed extension $\mathcal{J}$ of $\mathcal{I}$ has been obtained using only these operations, i.e. does not contain arbitrary (not obtained through typicality extension or minimal model completion) role edges. Therefore instead of guessing a maximal typicality model, we iteratively guess one typicality extension and compute its minimal model completion in polynomial time (in $|\Delta^{\mathcal{K}}|$) using Algorithm 3 until we reach a maximal typicality model.

Proposition 6.5. *For a minimal typicality model* $\mathcal{I} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{I}})$ *over the DKB* $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$*, Algorithm 4 terminates in polynomial time in* $|\Delta^{\mathcal{K}}|$ *for the inputs* $\mathcal{I}, \mathcal{K}$.

Proof. Every iteration of the while loop in Algorithm 4 strictly extends the current interpretation, i.e. $\mathcal{J}_i \subset \mathcal{J}_{i+1}$, by at least one role edge. That is, there exists an $r \in sig_R(\mathcal{K})$ such that $r^{\mathcal{J}_i} \subset r^{\mathcal{J}_{i+1}}$. Since there are only $|\Delta^{\mathcal{K}}|^2 * |sig_R(\mathcal{K})|$ distinct role edges possible in $\mathcal{I}$, such an extension can be constructed at most a polynomial number of times in $|\Delta^{\mathcal{K}}|$.

[13] As long as the size of $\Delta^{\mathcal{K}}$ is not sub-linear in $\|\mathcal{K}\|$.

Algorithm 4: Guess the construction of a maximal typicality model

Input: $\mathcal{I} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{I}})$: a minimal typicality model, $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$: a DKB
Output: A $\mathcal{J} \in typ^{\max}(\{\mathcal{I}\})$

1 $\mathcal{J}_0 := \mathcal{I}$
2 $i := 0$
3 **while** *$\mathcal{J}_i$ is typicality extensible* **do**
4 Guess a $\mathcal{J}'_i \in typ(\mathcal{J}_i)$ such that $mc(\mathcal{J}'_i) \neq \emptyset$
5 $\mathcal{J}_{i+1} := mmc(\mathcal{J}'_i)$
6 $i := i + 1$
7 **end**
8 $\mathcal{J}_{fin} := \mathcal{J}_i$
9 **return** $\mathcal{J}_{fin}$

Since computing a model completion (Line 5) has been shown to require polynomial time in $|\Delta^{\mathcal{K}}|$ (Proposition 6.1), it remains to show that the condition for the while loop can be checked in polynomial time in $|\Delta^{\mathcal{K}}|$. For every typicality interpretation $\mathcal{J}$, there are only polynomially many (in $|\Delta^{\mathcal{K}}|$) covering interpretations in $typ(\mathcal{J})$, i.e. the ones minimal w.r.t. $\subset$. For each one, Algorithm 3 can check in polynomial time whether they admit a model completion. If none of them does, it is clear that no other (even bigger) typicality extension admits a model completion. □

Proposition 6.5 shows that Algorithm 4 runs in polynomial time. Because it executes a non-deterministic guessing step, it is an NP algorithm. It remains to show that the minimal model completion that is finally returned by Algorithm 4 is actually a maximal typicality model of the input.

Proposition 6.6. *For a minimal typicality model $\mathcal{I} = (\Delta^{\mathcal{K}}, \cdot^{\mathcal{I}})$ over the DKB $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ and the interpretation $\mathcal{J}_{fin}$ as returned by Algorithm 4 over the inputs $\mathcal{I}, \mathcal{K}$, it holds that $\mathcal{J}_{fin} \in typ^{\max}(\{\mathcal{I}\})$.*

Proof. First of all, the guessing step in Line 4 is sound. The condition for the while loop ensures that an appropriate typicality extension exists for $\mathcal{J}_i$. After termination, $\mathcal{J}_{fin}$ is not typicality extensible, i.e. $\neg\exists\mathcal{J}' \in typ(\mathcal{J}_{fin}).mc(\mathcal{J}') \neq \emptyset$. In case $\mathcal{I}$ is not typicality extensible, $\mathcal{J}_{fin} = \mathcal{I}$ and clearly $\mathcal{J}_{fin} \in typ^{\max}(\{\mathcal{I}\})$.

It remains to show that there is a $k \geq 1$ s.t. $\mathcal{J}_{fin} \in T^k(\{\mathcal{I}\})$.

Let $n-1$ be the final iteration of Algorithm 4, i.e. $\mathcal{J}_{fin} = \mathcal{J}_n$. We show that for every $\mathcal{J}_i$ ($1 \leq i \leq n$), there exists a $k \geq 1$ s.t. $\mathcal{J}_i \in T^k(\{\mathcal{I}\})$ by induction on i. For $i = 1$, let $\mathcal{J}'_0$ be the guessed interpretation in $typ(\mathcal{I})$ with $mc(\mathcal{J}'_0) \neq \emptyset$. Applying the T-operator to a singleton set is deterministic and under the assumption that $\mathcal{I}$ is typicality extensible, we obtain the set $T(\{\mathcal{I}\}) = \{mmc(\mathcal{J}) \mid \mathcal{J} \in typ(\mathcal{I}) \wedge mc(\mathcal{J}) \neq \emptyset\}$, clearly showing that for the guessed $\mathcal{J}'_0$, $\mathcal{J}_1 = mmc(\mathcal{J}'_0) \in T^1(\{\mathcal{I}\})$. For the induction step, assume $\exists k \geq 1.\mathcal{J}_i \in T^k(\{\mathcal{I}\})$ and that $\mathcal{J}_i$ is typicality extensible. $\mathcal{J}_i$ has to be selected by the definition of the T-operator at some "time" $l > k$ *before* reaching the fixpoint of T. It is not hard to see that $T(\{\mathcal{J}_i\}) \subseteq T^l(\{\mathcal{I}\})$, thus the argument from the induction start shows that $\mathcal{J}_{i+1} = mmc(\mathcal{J}'_i) \in T(\{\mathcal{J}_i\}) \subseteq T^l(\{\mathcal{I}\})$. From the induction step, it directly follows that a $k \geq 1$ such that $\mathcal{J}_{fin} \in T^k(\{\mathcal{I}\})$ exists for $\mathcal{J}_{fin}$ as well. $\mathcal{J}_{fin}$ not being typicality extensible then implies that $\mathcal{J}_{fin} \in T^m(\{\mathcal{I}\})$ for all $m \geq k$, in particular $\mathcal{J}_{fin} \in typ^{\max}(\{\mathcal{I}\})$. □

Using Algorithm 4 we can describe a simple procedure for checking the complement of defeasible subsumption query entailment and defeasible instance checking. A defeasible subsumption query $C \sqsubsetapprox D$ is not entailed by a DKB $\mathcal{K}$ under rational (relevant) semantics *iff* $\exists\mathcal{J} \in typ^{\max}(\{\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rat}})\}).d^{\mathcal{D}_i}_C \notin D^{\mathcal{J}}$ for the smallest i with $d^{\mathcal{D}_i}_C \in \Delta^{\mathcal{K}}_{\text{rat}}$ ($\exists\mathcal{J} \in typ^{\max}(\{\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}})\}).d^{\mathcal{D}_C}_C \notin D^{\mathcal{J}}$ for relevant semantics, respectively). Likewise, an instance check $C(a)$ is not successful w.r.t. a DKB $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ under rational (relevant) semantics, *iff* $\exists\mathcal{J} \in typ^{\max}(\{\mathcal{I}(\Delta^{\mathcal{K}}) \cup \mathcal{I}_{\mathcal{A}',\mathcal{T}\cup\mathcal{T}_{\mathcal{D}}}\}).\mathcal{J} \not\models C(a)$ (rational: $\Delta^{\mathcal{K}} = \Delta^{\mathcal{K}}_{\text{rat}}$, $\mathcal{A}' = \widehat{\mathcal{A}}^s_{rat}$; relevant: $\Delta^{\mathcal{K}} = \Delta^{\mathcal{K}}_{\text{rel}}$, $\mathcal{A}' = \widehat{\mathcal{A}}^s_{rel}$).

After having established the general results for minimal model completion and maximising typicality based on the size of the typicality domain, we can investigate this size for both rational and relevant semantics and show the specific upper bound on both problems for defeasible instance checking and subsumption entailment.

6.1.1. Entailment under Rational Semantics

The problem of deciding rational entailment (defeasible subsumptions and instance checks under nested rational semantics) has as an input a DKB $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ and a defeasible subsumption query $\mathcal{K} \models^{(rat,nest)} C \sqsubsim D$ or instance check $\mathcal{K} \models^{(rat,nest)} C(a)$. Using the same argument as in [20] we can assume that C and D are syntactically contained in $\mathcal{K}$. Therefore we consider only the size of $\mathcal{K}$ in the investigations towards the complexity of this problem.

In the beginning of this article, we proved a claim from [4], showing that the complexity of computing the partition of the DBox for rational semantics resides in the same complexity class as deciding classical subsumption in the underlying description logic, even for $\mathcal{EL}_\bot$. Here, classical subsumption entailment can be decided in polynomial time, as we are restricted to $\mathcal{EL}_\bot$ [14].

Proposition 6.7. *The partition$(\mathcal{D}) = (E_1, \ldots, E_n)$ can be computed in polynomial time and $n \leq |\mathcal{D}|$.*

Using this result, we show that for a DKB $\mathcal{K}$ as the input (assume the query to be syntactically contained in $\mathcal{K}$), the size of the rational domain is polynomial in $\|\mathcal{K}\|$. This size determines the overall complexity of nested rational semantics based on the previously obtained general results. We essentially instantiate the previous results with a specific domain. .:

Proposition 6.8. *For a given DKB $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ and partition$(\mathcal{D}) = (E_1, \ldots, E_n)$, the size of the rational typicality domain $\Delta^{\mathcal{K}}_{rat} = \{d^{\mathcal{D}_i}_F \mid 1 \leq i \leq n, \mathcal{D}_i = \bigcup_{j=i}^{n} E_j, F \in Qc(\mathcal{K}), \overline{\mathcal{D}_i} \sqcap F \not\sqsubseteq_{\mathcal{T}} \bot\}$ is polynomial in $\|\mathcal{K}\|$ and can be computed in polynomial time in $\|\mathcal{K}\|$.*

Proof. It is not hard to see that $|\Delta^{\mathcal{K}}_{\text{rat}}| \leq |Qc(\mathcal{K})| * |partition(\mathcal{D})|$, where $|Qc(\mathcal{K})| \leq \|\mathcal{K}\|$ and $|partition(\mathcal{D})| \leq |\mathcal{D}|$, thus the size of $|\Delta^{\mathcal{K}}_{\text{rat}}|$ is at most quadratic in the size of the input. Furthermore, $\Delta^{\mathcal{K}}_{\text{rat}}$ can be computed using at most a polynomial number of classical entailment checks of the type $\overline{\mathcal{D}_i} \sqcap F \not\sqsubseteq_{\mathcal{T}} \bot$, each of which can be computed in polynomial time using Lemma 3.5 and [14]. □

We show the complexity upper bound for both defeasible instance and subsumption checking under nested rational semantics.

Theorem 6.9. *For a DKB $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$, entailment of a defeasible subsumption query $C \sqsubsim D$ and a defeasible instance check $C(a)$ can be decided in* co-NP*–Time under nested rational semantics.*

Proof. The minimal typicality model $\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rat}})$ can be constructed in polynomial time in $\|\mathcal{K}\|$ by doing a polynomial number of classical subsumption checks for a number of domain elements that is polynomial in $\|\mathcal{K}\|$ (Proposition 6.8). The ABox extension $\widehat{\mathcal{A}}^s_{rat}$ can be computed with at most $|sig_I(\mathcal{A})| * |partition(\mathcal{D})|$ (at most quadratic in $\|\mathcal{K}\|$) consistency checks (polynomial for $\mathcal{EL}_\bot$) and its size is at most $|\mathcal{A}| + (|sig_I(\mathcal{A})| * |\mathcal{D}|)$, i.e. quadratic in $|\mathcal{K}|$. Hence, the consistency checks are based on a knowledge base that is at most quadratic in the input DKB (c.f. Alg. 1). The extended TBox $\mathcal{T} \cup \mathcal{T}_{\mathcal{D}}$ is of size $|\mathcal{T}| + |\mathcal{D}|$ (linear in $|\mathcal{K}|$). Let $\mathcal{K}' = (\widehat{\mathcal{A}}^s_{rat}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}, \mathcal{D})$. $\mathcal{K}'$ is at most quadratic in $|\mathcal{K}|$. Additionally, $sig_R(\mathcal{K}') = sig_R(\mathcal{K})$, $sig_I(\widehat{\mathcal{A}}^s_{rat}) = sig_I(\mathcal{A})$ and $Qc(\mathcal{K}') = Qc(\mathcal{K})$ ($|sig_C(\mathcal{K}')| = |sig_C(\mathcal{K})| + |\mathcal{D}|$). The ABox interpretation $\mathcal{I}_{\widehat{\mathcal{A}}^s_{rat}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}}$ has a domain of size $|sig_I(\mathcal{A})| + |Qc(\mathcal{K})|$. Therefore, it requires at most

$$|sig_I(\mathcal{A})| * \big(|sig_C(\mathcal{K})| + |sig_R(\mathcal{K})| * |Qc(\mathcal{K})|\big)$$

many classical entailment checks (using $\mathcal{K}'$) to compute $\mathcal{I}_{\widehat{\mathcal{A}}^s_{rat}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}}$. To summarise, the minimal typicality model $\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rat}}) \cup \mathcal{I}_{\widehat{\mathcal{A}}^s_{rat}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}}$ can be computed in polynomial time in $\|\mathcal{K}\|$ and is of polynomial size in $\|\mathcal{K}\|$ (dominated by $\Delta^{\mathcal{K}}_{\text{rat}}$). A maximal typicality model of $\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rat}}) \cup \mathcal{I}_{\widehat{\mathcal{A}}^s_{rat}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}}$ can be constructed using Algorithm 4 in NP–Time by Proposition 6.5. Note, that model completions are constructed w.r.t. $(\mathcal{T}_{\mathcal{D}}, \mathcal{D})$, which is linear in the cardinality of $\mathcal{K}$. Finally, checking non-entailment of $C \sqsubsim D$ or $C(a)$ for a given typicality interpretation is linear in the size of the domain, hence providing an overall co-NP–Time procedure for deciding nested rational entailment. □

6.1.2. Entailment under Relevant Semantics

The main differences between nested rational and nested relevant semantics are the underlying typicality domains and the respective default assumption extensions of the ABox. The relevant typicality domain contains $\mathcal{O}(2^{\|\mathcal{K}\|})$ domain elements. Any of the discussed inferences requires to construct the minimal typicality model (empty ABox) over $\Delta^{\mathcal{K}}_{\text{rel}}$, which uses an exponential number of classical subsumption checks (in $\|\mathcal{K}\|$). The maximum number of possible role edges over the relevant domain is $\eta = |\Delta^{\mathcal{K}}_{\text{rel}}|^2 * |sig_R(\mathcal{K})|$, which, for an exponential number of domain elements, is double exponential in $\|\mathcal{K}\|$. A constructive enumeration of all maximal typicality models would therefore run in 2-EXP–Time. Alternatively, the guess and check approach can be used here as well to determine non-entailment of defeasible queries. Still, the construction of a maximal typicality model of exponential size (number of role edges) will be guessed using Algorithm 4. Luckily this does not result in a 2-EXP–Time procedure, but it requires to compute a minimal model completion (exponential in $\|\mathcal{K}\|$, for the exponential $\Delta^{\mathcal{K}}_{\text{rel}}$) up to η times. Deciding defeasible instance checks based on nested relevant semantics, requires always to compute the minimal typicality model over $\Delta^{\mathcal{K}}_{\text{rel}}$ first, as well as maximise a typicality interpretation that contains $\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}})$, i.e. is always at least as complex as deciding defeasible subsumption.

Proposition 6.10. *The relevant domain $\Delta^{\mathcal{K}}_{rel} = \{d^{\mathcal{U}}_F \mid \mathcal{U} \subseteq \mathcal{D}, \overline{\mathcal{U}} \sqcap F \not\sqsubseteq_{\mathcal{T}} \bot\}$ is of size exponential in $\|\mathcal{K}\|$ and can be computed in exponential time in $\|\mathcal{K}\|$.*

Proof. We have $|\Delta^{\mathcal{K}}_{\text{rel}}| \leq |Qc(\mathcal{K})| * 2^{|\mathcal{D}|}$, due to an analogous argument as in Proposition 6.8. The exponential comes from the fact that we allow representative domain elements for any subset of $\mathcal{D}$. Determining which subsets of $\mathcal{D}$ are consistent with every $F \in Qc(\mathcal{K})$ requires exponentially many classical subsumption checks, each of which can be determined in polynomial time. □

Theorem 6.11. *For a DKB $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$, defeasible entailment of a subsumption query $C \sqsubset\!\!\!\sim D$ and an instance check $C(a)$ can be decided in* co-NEXP*–Time under nested relevant semantics.*

Proof. The minimal typicality model $\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}})$ can be constructed in exponential time in $\|\mathcal{K}\|$ by doing a linear number of classical subsumption checks (P–time) for a number of domain elements exponential in $\|\mathcal{K}\|$ (Proposition 6.10). The extended ABox $\widehat{\mathcal{A}}^s_{rel}$ (Alg. 2) is constructed on top of $\widehat{\mathcal{A}}^s_{rat}$ (polynomial time and size in $|\mathcal{K}|$), using $|sig_I(\mathcal{A})|$ many extension steps that each require the computation of a maximal subset of $\mathcal{D}$ that is consistent with an individual, $\mathcal{D}_{(\mathcal{A}', \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}), a}$, which is exponential for $\mathcal{EL}_{\bot}$ knowledge bases $(\mathcal{A}', \mathcal{T} \cup \mathcal{T}_{\mathcal{D}})$. $\mathcal{A}'$ has the same size-bound as $\widehat{\mathcal{A}}^s_{rel}$ which in turn, is the same upper bound as for the size of $\widehat{\mathcal{A}}^s_{rat}$ (at most "all of the DBox" added for every individual). The remaining deliberations for $\mathcal{I}_{\widehat{\mathcal{A}}^s_{rel}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}}$ are analogous to the proof of Theorem 6.9. Thus $\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}}) \cup \mathcal{I}_{\widehat{\mathcal{A}}^s_{rel}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}}$ is of exponential size in $\|\mathcal{K}\|$, dominated by $\Delta^{\mathcal{K}}_{\text{rel}}$. A maximal typicality model of $\mathcal{I}(\Delta^{\mathcal{K}}_{\text{rel}}) \cup \mathcal{I}_{\widehat{\mathcal{A}}^s_{rel}, \mathcal{T} \cup \mathcal{T}_{\mathcal{D}}}$ can be constructed using Algorithm 4 in NEXP–Time by Proposition 6.5. Note, that model completions are constructed w.r.t. $(\mathcal{T}_{\mathcal{D}}, \mathcal{D})$, which is linear in the size of $\mathcal{K}$. Finally, checking non-entailment of $C \sqsubset\!\!\!\sim D$ or $C(a)$ for a given typicality interpretation is linear in the size of the domain (exponential), hence providing an overall co-NEXP–Time procedure for deciding nested relevant entailment. □

6.2. Lower Bound on Deciding Rational Entailments

We prove co-NP-hardness for deciding nested rational subsumption entailment by a reduction from (1-in-3)-positive 3sat, which is known to be NP–compl. [23]. Satisfiability of a (1-in-3)-positive 3sat formula is reduced to non-entailment of a defeasible subsumption query under nested rational semantics, hence the co-NP-hardness. We consider only non-entailment of defeasible subsumption at first, and add a comment to how this reduction proof applies to non-entailment of defeasible instance checking under nested rational semantics in the end.

*The reduction from (1-in-3)-positive 3*SAT. A (1-in-3)-positive 3SAT problem is given with a propositional formula φ in conjunctive normal form with clauses of size 3. The general representation for a (1-in-3)-positive 3SAT problem with n clauses and k propositional variables $\mathcal{V} = \{x_1, \dots, x_k\}$ is $\varphi = \bigwedge_{i=1}^{n}(x_{i_1}, x_{i_2}, x_{i_3})$ s.t. $i_1, i_2, i_3 \in \{1, \dots, k\}$ $(i_1 \neq i_2 \neq i_3)$ for all $1 \leq i \leq n$. $(x_{i_1}, x_{i_2}, x_{i_3})$ is called a *clause* in φ. Such an instance φ is satisfied by a truth assignment if in every clause in φ there is exactly one propositional variable that is assigned to true (note, all variables occur as a positive literal). More formally, an assignment for a formula φ is a function $\sigma : \mathcal{V} \to \{\top, \bot\}$ that assigns *every* propositional variable occurring in φ either $\top$ or $\bot$. σ is extended to apply to a (1-in-3)-positive 3SAT formula s.t. $\widehat{\sigma}(\varphi) = \top$ *iff* for all $i \in \{1, \dots, n\}$ there is *exactly* one x_{i_j} $(j \in \{1, 2, 3\})$ s.t. $\sigma(x_{i_j}) = \top$. Satisfiability for (1-in-3)-positive 3SAT is NP–compl. [23] and thus, if successfully reduced to non-entailment of defeasible subsumption, shows CO-NP-hardness of our decision problem.

Given φ an instance of (1-in-3)-positive 3SAT, we reduce satisfiability to unsatisfiability of a query under nested rational entailment. We construct the defeasible knowledge base $\mathcal{K}_\varphi$ with

$$N_C = \{A, B, X\} \cup \{C_i \mid 1 \leq i \leq n\} \tag{2}$$
$$N_R = \{s, r_1, \dots, r_k\} \tag{3}$$

where k is the number of distinct propositional variables occurring in φ and n is the number of clauses occurring in φ. W.l.o.g. we assume a linear order on the clauses in φ, simply to reference them by indices $1 \leq i \leq n$. Domain elements that are referenced in the following occur in the rational domain, which is shown after the full construction of $\mathcal{K}_\varphi$. The main idea is to let upgrades of role edges r_j correspond to the assignment of the propositional variable x_j. $\sigma(x_j) = \top$ translates to a specific r_j edge $(d^{\emptyset}_A, d^{\emptyset}_B)$ to be upgraded to $(d^{\emptyset}_A, d^{\mathcal{D}}_B)$, where for a satisfying assignment σ, our specific query is not entailed. The DBox shall be rather small

$$\mathcal{D} = \{\top \mathrel{\sqsubset\!\!\!\sim} X\} \tag{4}$$

while the remainder of the reduction can be achieved using the TBox and the query. It is easy to see that the partition of the DBox according to [4] contains only $E_1 = \mathcal{D}$, hence the chain of decreasing DBox subsets $\mathcal{D}_0 = \mathcal{D}$ and $\mathcal{D}_1 = \emptyset$. This means that the resulting typicality interpretations has two levels of typicality. The structure of the resulting minimal typicality model $\mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\text{rat}})$ is mostly influenced by

$$\mathcal{T}_{struct} = \{\exists s.A \sqsubseteq \top, \tag{5}$$
$$A \sqcap X \sqsubseteq \bot, \tag{6}$$
$$A \sqsubseteq \exists s.B, \tag{7}$$
$$A \sqsubseteq \exists r_1.B \sqcap \dots \sqcap \exists r_k.B\} \tag{8}$$

where (5) ensures that the representative $d^{\emptyset}_A$ occurs in $\Delta^{\mathcal{K}_\varphi}_{\text{rat}}$, (6) is optional but makes things easier later on by ensuring that $d^{\mathcal{D}}_A \notin \Delta^{\mathcal{K}_\varphi}_{\text{rat}}$. (7) is required "in the end", to ensure that the query is not entailed *iff* φ is satisfiable and (8) prepares the maximisation procedure by introducing all r_i edges from $d^{\emptyset}_A$ to $d^{\emptyset}_B$ into $\mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\text{rat}})$. We continue with two kinds of TBoxes capturing information about each clause of φ.

$$\mathcal{T}^i_{const} = \{\exists r_{i_1}.X \sqcap \exists r_{i_2}.X \sqsubseteq \bot, \tag{9}$$
$$\exists r_{i_2}.X \sqcap \exists r_{i_3}.X \sqsubseteq \bot, \tag{10}$$
$$\exists r_{i_1}.X \sqcap \exists r_{i_3}.X \sqsubseteq \bot\} \tag{11}$$

and

$$\mathcal{T}^i_{clause} = \{\exists r_{i_1}.X \sqsubseteq C_i, \tag{12}$$
$$\exists r_{i_2}.X \sqsubseteq C_i, \tag{13}$$
$$\exists r_{i_3}.X \sqsubseteq C_i\ \} \tag{14}$$

where $i_1, i_2, i_3 \in \{1, \ldots, k\}$ and the indices correspond to those from the i-th clause in φ. $\mathcal{T}^i_{const}$ describes the disjointness constraints for the typicality upgrade procedure. For instance the clause (x_1, x_3, x_5) prohibits any pair of these three variables to be set to $\top$ at the same time. Likewise the constraints $\exists r_1.X \sqcap \exists r_3.X \sqsubseteq \bot$, $\exists r_3.X \sqcap \exists r_5.X \sqsubseteq \bot$, $\exists r_1.X \sqcap \exists r_5.X \sqsubseteq \bot$ prohibit two role upgrades of the roles r_1, r_3, r_5 to occur at the same time, since the typical concept representatives satisfy $\top \mathrel{\sqsubseteq\!\!\!\!\!\sim} X$. The TBox $\mathcal{T}^i_{clause}$ creates a kind of marker at the domain element $d^{\emptyset}_A$. For a maximal typicality model $\mathcal{I}$, $d^{\emptyset}_A \in C^{\mathcal{I}}_i$ holds *iff* for clause i in φ at least one of its variables is assigned $\top$. Together $\mathcal{T}^i_{const}$ and $\mathcal{T}^i_{clause}$ ensure that $C_1, \ldots, C_n$ are satisfied by $d^{\emptyset}_A$ when every clause in φ has at least one and at most one (i.e. exactly one) variable set to $\top$ (i.e. role upgraded to contain $(d^{\emptyset}_A, d^{\mathcal{D}}_B)$).

The role edge $(d^{\emptyset}_A, d^{\emptyset}_B) \in s^{\mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\mathrm{rat}})}$ is used to invert the query entailment, to be explicit, the query is

$$A \mathrel{\sqsubseteq\!\!\!\!\!\sim} \exists s.X \tag{15}$$

and the inverting is ensured by the following disjointness constraint

$$\mathcal{T}_Q = \{\exists s.X \sqcap C_1 \sqcap \cdots \sqcap C_n \sqsubseteq \bot\}. \tag{16}$$

$\mathcal{T}_Q$ ensures that the s successor $(d^{\emptyset}_A, d^{\emptyset}_B)$ can only be upgraded if *not* all clauses are satisfied. Overall, we define

$$\mathcal{K}_\varphi = (\mathcal{T}_{struct} \cup \bigcup_{i=1}^{n} (\mathcal{T}^i_{const} \cup \mathcal{T}^i_{clause}) \cup \mathcal{T}_Q, \mathcal{D}). \tag{17}$$

As stated before, $C_1, \ldots, C_n$ are satisfied by $d^{\emptyset}_A$ in a maximal typicality model that corresponds to a satisfying assignment of the variables in φ. All C_is being satisfied at $d^{\emptyset}_A$ then prohibits the edge $(d^{\emptyset}_A, d^{\emptyset}_B) \in s^{\mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\mathrm{rat}})}$ to be upgraded, resulting in non-entailment of the query. Of course the s edge could be upgraded prematurely, simply prohibiting some r_j upgrade, however to show non-entailment, the *existence* of one maximal typicality model as a counterexample is sufficient.

Complexity and Correctness of the Reduction. We proceed by showing that this reduction is linear and that satisfiability of φ corresponds to non-entailment of the constructed query under nested rational semantics.

Proposition 6.12. *$\|\mathcal{K}_\varphi\|$ is*

1. *linear in $|\varphi|$ and*
2. *can be constructed in linear time (in $|\varphi|$).*

Proof. For $\varphi = \bigwedge_{i=1}^{n}(x_{i_1}, x_{i_2}, x_{i_3})$, $|\varphi| = 3 * n$. For the constant part in $\mathcal{K}_\varphi$, note that $\|\mathcal{D}\| = 2$ and the query $\|A \mathrel{\sqsubseteq\!\!\!\!\!\sim} \exists s.X\| = 3$. The TBox parts $\mathcal{T}_{struct}$ and $\mathcal{T}_Q$ are linear in $|\varphi|$, as $\|\mathcal{T}_{struct}\| = 2k + 10$ for k, the amount of propositional variables used in φ and $\|\mathcal{T}_Q\| = n + 3$. The TBox parts $\mathcal{T}^i_{const}$ and $\mathcal{T}^i_{clause}$ have the constant size $\|\mathcal{T}^i_{const}\| = 15$ and $\|\mathcal{T}^i_{clause}\| = 9$ for every i. Therefore the total size of $\mathcal{T}$ amounts to $\|\mathcal{T}\| = 2k + 10 + n \cdot (15 + 9) + n + 3 = 22n + 2k + 13$ which (since $k \leq |\varphi|$) is clearly linear in $|\varphi|$.

The algorithm constructing $\mathcal{K}_\varphi$ needs to iterate over every clause in φ exactly once, therefore covering the second claim. □

We continue to prove the main claim required for the desired hardness result.

Lemma 6.13. *Formula φ is satisfiable iff $\mathcal{K}_\varphi \not\models^{(rat,nest)} A \mathrel{\sqsubseteq\!\!\!\!\!\sim} \exists s.X$.*

Proof. For $\mathcal{K}_\varphi = (\mathcal{T}, \mathcal{D})$ it is obvious that *partition*$(\mathcal{D}) = (\mathcal{D})$, hence we investigate two levels of typicality and domain elements satisfy either $\mathcal{D}$ or $\emptyset$. Clearly, $d^{\mathcal{D}}_A \notin \mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\mathrm{rat}})$ (since $A \sqcap (\neg\top \sqcup X) \sqsubseteq_{\mathcal{T}_{struct}} \bot$). Thus, $d^{\emptyset}_A$ is the most typical concept representative of A (i.e. least $i \in \{1, 2\}$ for $\mathcal{D}_1 = \mathcal{D}$, $\mathcal{D}_2 = \emptyset$ s.t. $d^{\mathcal{D}_i}_C \in \Delta^{\mathcal{K}_\varphi}_{\mathrm{rat}}$) and therefore the right-hand side of the equivalence in this lemma is determined by investigating the element $d^{\emptyset}_A$, i.e.

$$\exists \mathcal{J} \in typ^{\mathsf{max}}(\{\mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\mathrm{rat}})\}).d^{\emptyset}_A \notin (\exists s.X)^{\mathcal{J}}. \tag{18}$$

To prove this lemma, we show both directions,

(i) If φ is satisfiable then (18)

(ii) If (18) then there is a satisfying assignment for φ

$\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi})$ is uniquely determined, it has the following definitions due to the construction of $\mathcal{K}_\varphi$ (note $Qc(\mathcal{K}_\varphi) = \{A, B, X\}$) and it is depicted in Figure 4:

- $\Delta_{\text{rat}}^{\mathcal{K}_\varphi} = \{d_X^\emptyset, d_X^\mathcal{D}, d_A^\emptyset, d_B^\emptyset, d_B^\mathcal{D}\}$
- $A^{\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi})} = \{d_A^\emptyset\}$
- $B^{\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi})} = \{d_B^\emptyset, d_B^\mathcal{D}\}$
- $X^{\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi})} = \{d_X^\emptyset, d_X^\mathcal{D}, d_B^\mathcal{D}\}$
- $t^{\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi})} = \{(d_A^\emptyset, d_B^\emptyset)\}$ $(t \in \{s, r_1, \ldots, r_k\})$

One can verify that the minimal typicality model for $\mathcal{K}_\varphi = (\mathcal{T}, \mathcal{D})$ is the same as that for $\mathcal{K}' = (\mathcal{T}_{struct}, \mathcal{D})$ and due to the simple nature of $\mathcal{T}_{struct}$ it is not hard to check the respective extensions of concept and role names above to be correct.

All interpretations in $typ^{\mathsf{max}}(\{\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi})\})$ extend $\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi})$ and therefore have the same domain. Thus, we consider only this domain in the following. Towards proving (i), assume the assignment σ to satisfy φ. We define several "partial" interpretations that, when combined (union), play certain roles. We call them partial because they only contain the difference between the minimal typicality model for $\mathcal{K}_\varphi$ and some extension of it exhibiting desired features, as such they are not models of $\mathcal{K}_\varphi$ themselves.

We begin with $\mathcal{J}_{t1}^\sigma = (\Delta_{\text{rat}}^{\mathcal{K}_\varphi}, \cdot^{\mathcal{J}_{t1}^\sigma})$ such that

- $r_i^{\mathcal{J}_{t1}^\sigma} = \begin{cases} \{(d_A^\emptyset, d_B^\mathcal{D})\} & \text{if } \sigma(x_i) = \top \\ \emptyset & \text{otherwise} \end{cases}$

  for $1 \le i \le k$ (for an example see Figure 5). Everything in $sig(\mathcal{K}_\varphi)$ that not explicitly defined, is interpreted as $\emptyset$, which is also the case for the remaining interpretation constructions in this proof.

It is easy to see that $\mathcal{J}_{t1}^\sigma$ contains only role edges that belong to $TR_{\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi})}(r_i)$ for some $r_i \in sig_R(\mathcal{K})$ (c.f. Definition 4.22) and therefore $\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi}) \cup \mathcal{J}_{t1}^\sigma \in typ(\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi}))$ clearly holds. Furthermore, if σ is a satisfying assignment for φ, for each clause $(x_{j_1}, x_{j_2}, x_{j_3})$ $(j_1, j_2, j_3 \in \{1, \ldots, k\})$ *exactly one* of the three roles $r_{j_1}, r_{j_2}, r_{j_3}$ have a edge from $d_A^\emptyset$ to $d_B^\mathcal{D}$ in $\mathcal{J}_{t1}^\sigma$. The second "partial" interpretation we are using is $\mathcal{J}_{mmc}^\sigma = (\Delta_{\text{rat}}^{\mathcal{K}_\varphi}, \cdot^{\mathcal{J}_{mmc}^\sigma})$ with

- $C_j^{\mathcal{J}_{mmc}^\sigma} = \{d_A^\emptyset\}$ for all clause indices $1 \le j \le n$

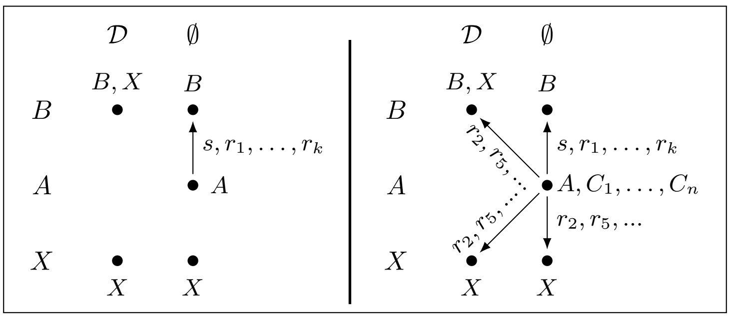

Figure 4: Minimal typicality model $\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi})$ (left) and (an example of) the maximal typicality model $\mathcal{J}_\sigma$ (right)

- $r_i^{\mathcal{J}^\sigma_{mmc}} = \begin{cases} \{(d^\emptyset_A, d^\emptyset_X)\} & \text{if } \sigma(x_i) = \top \\ \emptyset & \text{otherwise} \end{cases}$, for all variable indices $1 \le i \le k$.

$\mathcal{J}^\sigma_{mmc}$ exhibits two features, it satisfies the concept C_j for the j-th clause in φ with $d^\emptyset_A$ for all clauses (because σ is assumed to satisfy φ). Secondly, $\mathcal{J}^\sigma_{mmc}$ contains the edge $(d^\emptyset_A, d^\emptyset_X)$ for exactly the same roles as were used in $\mathcal{J}^\sigma_{t1}$. We show that $\mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\text{rat}}) \cup \mathcal{J}^\sigma_{t1} \cup \mathcal{J}^\sigma_{mmc} = mmc(\mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\text{rat}}) \cup \mathcal{J}^\sigma_{t1})$. Condition 1 of Definition 4.23 is trivially satisfied, i.e. $\mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\text{rat}}) \cup \mathcal{J}^\sigma_{t1} \subseteq \mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\text{rat}}) \cup \mathcal{J}^\sigma_{t1} \cup \mathcal{J}^\sigma_{mmc}$. It is not hard to check whether $\mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\text{rat}}) \cup \mathcal{J}^\sigma_{t1} \cup \mathcal{J}^\sigma_{mmc}$ is a standard typicality interpretation. Condition 1 of Definition 4.3 (standard interpretation) holds because of $\mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\text{rat}}) \subseteq \mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\text{rat}}) \cup \mathcal{J}^\sigma_{t1} \cup \mathcal{J}^\sigma_{mmc}$ and Proposition 4.4. The only role edges that have to be inspected for checking Condition 2 of the standard property are those added by $\mathcal{J}^\sigma_{t1}$, since the others already exist in $\mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\text{rat}})$ which is known to be standard. Each edge $(d^\emptyset_A, d^\mathcal{D}_B)$ for some r_j in $\mathcal{J}^\sigma_{t1}$ results in $d^\emptyset_A \in (\exists r_j.X)^{\mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\text{rat}}) \cup \mathcal{J}^\sigma_{t1}}$. Since $\mathcal{J}^\sigma_{mmc}$ contributes for each such edge exactly one new edge $(d^\emptyset_A, d^\emptyset_X)$, Condition 2 of Definition 4.3 is clearly satisfied. It remains to show that $\mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\text{rat}}) \cup \mathcal{J}^\sigma_{t1} \cup \mathcal{J}^\sigma_{mmc}$ satisfies $\mathcal{K}_\varphi$ and that it is minimal w.r.t. all those properties. The DBox subsets are clearly still satisfied for the respective domain elements and $\mathcal{T}_{struct}$ is satisfied as well since the extensions of A and X remain as in $\mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\text{rat}})$. To verify that $\mathcal{T}^i_{clause}$ is satisfied for every clause index $1 \le i \le n$, we can see that

$$\begin{aligned} & d^\emptyset_A \in (\exists r_{i_j}.X)^{\mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\text{rat}}) \cup \mathcal{J}^\sigma_{t1} \cup \mathcal{J}^\sigma_{mmc}} \\ \textit{iff } & (d^\emptyset_A, d^\mathcal{D}_B) \in r_{i_j}^{\mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\text{rat}}) \cup \mathcal{J}^\sigma_{t1} \cup \mathcal{J}^\sigma_{mmc}} \\ \textit{iff } & \sigma(x_{i_j}) = \top \text{ by the construction of } \mathcal{J}^\sigma_{t1}. \end{aligned} \tag{$*$}$$

Under the assumption that σ satisfies φ, this holds for exactly one r_{i_j} $(1 \le j \le 3)$ per clause index i, thus $d^\emptyset_A$ is required to belong to the extension of each C_i $(1 \le i \le n)$, which it does by the construction of $\mathcal{J}^\sigma_{mmc}$. For $\mathcal{T}^i_{const}$, $d^\emptyset_A \in (\exists r_{i_j}.X \sqcap \exists r_{i_l}.X)^{\mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\text{rat}}) \cup \mathcal{J}^\sigma_{t1} \cup \mathcal{J}^\sigma_{mmc}}$ holds with an analogous condition as for $(*)$, which, for σ satisfying φ, can never be the case for any clause index $1 \le i \le n$ and $(j, l) \in \{(1, 2), (1, 3), (2, 3)\}$ because otherwise there would be a clause with more than 1 variable mapped to $\top$ under σ. $\mathcal{T}_Q$ remains satisfied because $d^\emptyset_A$ has no s successor to an element belonging to the extension of X in $\mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\text{rat}}) \cup \mathcal{J}^\sigma_{t1} \cup \mathcal{J}^\sigma_{mmc}$. Also, no other domain element belongs to the extension of any C_i.

In order to check for minimality of $\mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\text{rat}}) \cup \mathcal{J}^\sigma_{t1} \cup \mathcal{J}^\sigma_{mmc}$ among $mc(\mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\text{rat}}) \cup \mathcal{J}^\sigma_{t1})$, we can only attempt to remove role edges or $d^\emptyset_A$ from C_is introduced with $\mathcal{J}^\sigma_{mmc}$ because otherwise Condition 1 of Def. 4.23 would be violated. If we removed the edge $(d^\emptyset_A, d^\emptyset_X)$ from the extension of some r_i $(1 \le i \le k)$ then we know that for that i, $d^\emptyset_A \in (\exists r_i.X)^{\mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\text{rat}}) \cup \mathcal{J}^\sigma_{t1}}$ due to the construction of $\mathcal{J}^\sigma_{t1}$. It follows that the standard property would not be satisfied for the reduced interpretation. Likewise, removing $d^\emptyset_A$ from the extension of some C_j $(1 \le j \le n)$ results in the fact that $\mathcal{T}^j_{clause}$ is not satisfied because $d^\emptyset_A \in (\exists r_{j_l}.X)^{\mathcal{I}(\Delta^{\mathcal{K}_\varphi}_{\text{rat}}) \cup \mathcal{J}^\sigma_{t1}}$ for exactly one $1 \le l \le 3$ due to the construction of $\mathcal{J}^\sigma_{t1}$ and the fact that σ satisfies φ.

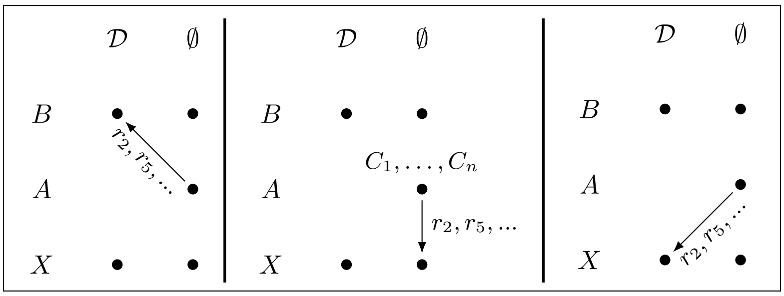

Figure 5: Examples for (from left to right) $\mathcal{J}^\sigma_{t1}$, $\mathcal{J}^\sigma_{mmc}$ and $\mathcal{J}^\sigma_{t2}$

It can be readily seen that $\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi}) \cup \mathcal{J}_{t1}^\sigma \cup \mathcal{J}_{mmc}^\sigma$ is typicality extensible, hence we define $\mathcal{J}_{t2}^\sigma = (\Delta_{\text{rat}}^{\mathcal{K}_\varphi}, \cdot^{\mathcal{J}_{t2}^\sigma})$ with

- $r_i^{\mathcal{J}_{t2}^\sigma} = \begin{cases} \{(d_A^\emptyset, d_X^{\mathcal{D}})\} & \text{if } \sigma(x_i) = \top \\ \emptyset & \text{otherwise} \end{cases}$
  for all $1 \leq i \leq k$.

$\mathcal{J}_{t2}^\sigma$ contains exactly those role edges upgrading the edges $(d_A^\emptyset, d_X^\emptyset)$ in $\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi}) \cup \mathcal{J}_{t1}^\sigma \cup \mathcal{J}_{mmc}^\sigma$. Let $\mathcal{J}_\sigma = \mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi}) \cup \mathcal{J}_{t1}^\sigma \cup \mathcal{J}_{mmc}^\sigma \cup \mathcal{J}_{t2}^\sigma$. With the construction of $\mathcal{J}_{t2}^\sigma$, is easy to see that $\mathcal{J}_\sigma \in typ(\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi}) \cup \mathcal{J}_{t1}^\sigma \cup \mathcal{J}_{mmc}^\sigma)$. Since $d_X^\emptyset$ belongs to exactly the same extensions as $d_X^{\mathcal{D}}$ (in all of the discussed interpretations $\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi})$, $\mathcal{J}_{t1}^\sigma$, $\mathcal{J}_{t2}^\sigma$, $\mathcal{J}_{mmc}^\sigma$), it can be seen that for all concepts C with $Qc(C) \subseteq Qc(\mathcal{K}_\varphi)$, the extensions in $\mathcal{J}_\sigma$ coincide with those in $\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi}) \cup \mathcal{J}_{t1}^\sigma \cup \mathcal{J}_{mmc}^\sigma$. Since $\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi}) \cup \mathcal{J}_{t1}^\sigma \cup \mathcal{J}_{mmc}^\sigma$ satisfies Conditions 2 and 3 of Definition 4.23 (model completion), so does $\mathcal{J}_\sigma$, which implies $\mathcal{J}_\sigma \in mc(\mathcal{J}_\sigma)$ which immediately implies $\mathcal{J}_\sigma = mmc(\mathcal{J}_\sigma)$. With the construction of $\mathcal{J}_\sigma$ and the definition of the T–operator, we can see that there must be some $l \geq 2$ such that $\mathcal{J}_\sigma \in T^l(\{\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi})\})$. By showing that $\mathcal{J}_\sigma$ is not typicality extensible, we would have shown that it remains in $T^m(\{\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi})\})$ for all $m \geq l$ and thus $\mathcal{J}_\sigma \in typ^{\max}(\{\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi})\})$. Observe that $TR_{\mathcal{J}_\sigma}(s) = TR_{\mathcal{J}_\sigma}(r_i) = \{(d_A^\emptyset, d_B^{\mathcal{D}})\}$ for all those $1 \leq i \leq k$ where $\sigma(x_i) = \bot$. It is enough to check for any single potential typicality upgrade whether it admits a model completion. W.l.o.g. we investigate $\mathcal{J}' = \mathcal{J}_\sigma[r_i/r_i^{\mathcal{J}_\sigma} \cup \{(d_A^\emptyset, d_B^{\mathcal{D}})\}]$ for any (single) $1 \leq i \leq k$ s.t. $\sigma(x_i) = \bot$. In φ there exists some clause $1 \leq l \leq n$ where x_i occurs together with some x_j such that $\sigma(x_j) = \top$, for σ satisfying φ. Since the edge $(d_A^\emptyset, d_B^{\mathcal{D}})$ exists in $r_j^{\mathcal{J}_\sigma}$ by the construction of $\mathcal{J}_{t1}^\sigma$, such an upgrade would therefore have the effect that $d_A^\emptyset \in (\exists r_i.X \sqcap \exists r_j.X)^{\mathcal{J}'}$ which makes it impossible for $\mathcal{J}'$ and any extension of $\mathcal{J}'$ to satisfy $\mathcal{T}_{const}^l$. Extending $\mathcal{J}_\sigma$ to $\mathcal{J}'' = \mathcal{J}_\sigma[s := s^{\mathcal{J}_\sigma} \cup \{(d_A^\emptyset, d_B^{\mathcal{D}})\}]$ results in the fact that $d_A^\emptyset \in (\exists s.X)^{\mathcal{J}''}$ which together with the already established property of $\mathcal{J}_\sigma$ that $d_A^\emptyset$ belongs to the extension of every C_i ($1 \leq i \leq n$) makes it impossible for $\mathcal{J}''$ or any extension of $\mathcal{J}''$ to satisfy $\mathcal{T}_Q$. Concluding the proof for (i), we have shown that $\mathcal{J}_\sigma \in typ^{\max}(\{\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi})\})$ and for the most typical representative of A, i.e. $d_A^\emptyset$, it is obvious that $d_A^\emptyset \notin (\exists s.X)^{\mathcal{J}_\sigma}$, thus we have effectively constructed a counterexample to the entailment of the query $A \mathrel{\underset{\sim}{\sqsubset}} \exists s.X$.

For showing (ii), assume $\exists \mathcal{J} \in typ^{\max}(\{\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi})\}).d_A^\emptyset \notin (\exists s.X)^{\mathcal{J}}$ and construct the assignment $\sigma_{\mathcal{J}}$ from $\mathcal{J}$ with

$$\sigma_{\mathcal{J}}(x_i) = \top \textit{ iff } (d_A^\emptyset, d_B^{\mathcal{D}}) \in r_i^{\mathcal{J}} \tag{19}$$

for all $1 \leq i \leq k$. We continue the proof by contradiction. The assignment $\sigma_{\mathcal{J}}$ does not satisfy φ *iff* there is a clause in φ with index $l \in \{1, \ldots, n\}$ such that

(a) $\sigma_{\mathcal{J}}(x_{l_1}) = \sigma_{\mathcal{J}}(x_{l_2}) = \sigma_{\mathcal{J}}(x_{l_3}) = \bot$, or

(b) two or more of the variables $x_{l_1}, x_{l_2}, x_{l_3}$ are mapped to $\top$ by $\sigma_{\mathcal{J}}$.

First, observe that $(d_A^\emptyset, d_B^{\mathcal{D}}) \notin s^{\mathcal{J}}$ (otherwise, $\mathcal{J}$ would satisfy $A \mathrel{\underset{\sim}{\sqsubset}} \exists s.X$) and no typicality extension of $\mathcal{J}$ can be completed into a model because otherwise $\mathcal{J}$ would not be maximal, especially $\mathcal{J}' = \mathcal{J}[s/s^{\mathcal{J}} \cup \{(d_A^\emptyset, d_B^{\mathcal{D}})\}]$. The reason for $mc(\mathcal{J}') = \emptyset$ can only be $d_A^\emptyset \in (\exists s.X \sqcap C_1 \sqcap \cdots \sqcap C_n)^{\mathcal{J}'}$ (i.e. $\mathcal{J}' \not\models \mathcal{T}_Q$), which implies $d_A^\emptyset \in C_l^{\mathcal{J}}$ for all $1 \leq l \leq n$. Assuming there is a clause index l for which σ has the property proposed in (a), it is not hard to see that $\mathcal{J}[C_l/\emptyset]$ still satisfies $\mathcal{T}_{clause}^l$ since $d_A^\emptyset \notin (\exists r_{l_m}.X)^{\mathcal{J}}$ for $m \in \{1, 2, 3\}$, contradicting that $\mathcal{J}$ was reached with the T–operator (i.e. the typicality extension of $\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi})$ with the exact same set of role edges as in $\mathcal{J}$, has a minimal model completion that is strictly smaller than $\mathcal{J}$), hence contradicting $\mathcal{J} \in typ^{\max}(\{\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi})\})$.

For (b) it follows immediately that $\mathcal{J}$ cannot be a model of $\mathcal{T}_{const}^l$ because $d_A^\emptyset \in (\exists r_{l_m}.X \sqcap \exists r_{l_o}.X)^{\mathcal{J}}$ (for some $1 \leq m < o \leq 3$), again contradicting $\mathcal{J} \in typ^{\max}(\{\mathcal{I}(\Delta_{\text{rat}}^{\mathcal{K}_\varphi})\})$.

Since both (a) and (b) have lead to the contradiction of $\mathcal{J}$ being a maximal typicality model, it follows that $\sigma_{\mathcal{J}}$ is a satisfying assignment for φ, thus concluding the proof of (ii). □

	Rational	Relevant
Propositional	P–compl.	in EXP
Nested	CO-NP–compl.	in CO-NEXP

Table 1: $\mathcal{EL}_\bot$ complexity results for reasoning under the DDL semantics characterised by $\{rat, rel\} \times \{prop, nest\}$.

Consider a slight change in $\mathcal{K}_\varphi$ where $\exists s.A \sqsubseteq \top$ is not contained in $\mathcal{T}_{struct}$ and the ABox $\mathcal{A} = \{A(a)\}$ is added. Immediately, the representative $d_A^\emptyset$ will not be contained in the rational domain anymore. It is possible to imagine how the domain element d_a in $\mathcal{I}(\Delta_{\mathrm{rat}}^{\mathcal{K}_\varphi}) \cup \mathcal{I}_{\mathcal{A},\mathcal{T}}$, belonging to the extension of A in the minimal typicality model, now behaves exactly as the element $d_A^\emptyset$ did before. Therefore, non-entailment of the defeasible instance check $(\exists s.X)(a)$ would correspond to satisfiability of φ.[14]

Theorem 6.14. *Deciding nested rational entailment (subsumption and instance checking) is* CO-NP*–hard.*

PROOF. If it would be possible to solve the complement of nested rational entailment in less than NP–Time, Lemma 6.13 implies that we would have a faster way of solving (1-in-3)-positive 3SAT which contradicts its NP–completeness. The proof of Lemma 6.13 is easily adapted to defeasible instance checking. If the complement of nested rational entailment is NP-hard, then nested rational entailment is of course CO-NP-hard. □

As a consequence of Theorem 6.9 and Theorem 6.14 we have the following corollary and one of the main results of this article.

Corollary 6.15. *Deciding nested rational entailment is* CO-NP*–compl..*

As a result of this section, we present the overview of known complexities for deciding defeasible entailments (subsumption and instance checking alike) in $\mathcal{EL}_\bot$ w.r.t. the discussed semantics as characterised by pairs from $\{rat, rel\} \times \{prop, nest\}$ throughout the article in Table 1. As usual, the "in" notation expresses an upper bound and "-c" denotes completeness. The completeness for propositional rational reasoning is shown in Section 3 as a confirmation to the claim in [4]. The upper bound on propositional relevant reasoning comes from the worst-case exponential number of justifications [21] that have to be enumerated to determine consistent DBox subsets. Lower bounds for propositional (materialisation-based) relevant reasoning in $\mathcal{EL}_\bot$ have not been discussed in the literature, to the best of our knowledge. The complexity results for reasoning under nested semantics are new and have been shown in this article. Discovering the precise lower bound on nested relevant reasoning remains a non-trivial and open problem.

7. Conclusion and Future Work

In this paper we have investigated the defeasible Description Logic $\mathcal{EL}_\bot$ for the reasoning problems of subsumption and instance checking. Starting from earlier work on the same DDL, we have illustrated the issue of neglecting *all* defeasible information for quantified concepts that occurs from materialisation-based approaches. The prominence of rational reasoning in DDLs achieved by materialisation-based algorithms and the adaptation of the propositional KLM postulates to DLs, provides a solid motivation to investigate solutions that are appropriate for DLs as they resolve the mentioned issue regarding quantification. We initially had introduced the typicality model approach to alleviate the insufficiency of materialisation for defeasible subsumption under rational and relevant closure in [11, 12].

New contributions of this article are characterisations of standard inferences under non-monotonic semantics for defeasible Description Logics w.r.t. two parameters, namely different strength and coverage.

[14] Even though in classical reasoning, a lower bound on subsumption checking immediately translates to a lower bound on instance checking, such a consequence is not trivial in DDL semantics.

This includes the investigation of existing materialisation-based algorithms (subsumption, instance checking [4]) as well as the new reasoning task: defeasible instance checking under relevant semantics which has not been investigated so far. We showed for both reasoning problems that semantics of propositional coverage coincide with materialisation-based approaches and that semantics with nested coverage of defeasible information properly extend the sets of obtained inferences. Furthermore, nested coverage provides conclusions regarding defeasible information for quantified concepts that were criticised to be missing under materialisation. For a thorough characterisation of the discussed semantics, we presented complexity results for defeasible subsumption and instance checking under propositional and nested, rational and relevant semantics.

As an initial result, we provided a materialisation-based reduction to classical reasoning in the tractable sub-Boolean DL $\mathcal{EL}_\bot$. This certifies that propositional coverage need not become intractable for $\mathcal{EL}_\bot$ as conjectured in [4]. Tractability of computing propositional rational closure in defeasible $\mathcal{EL}_\bot$ has been independently and in parallel investigated by the authors Casini, Meyer and Straccia since 2015. The first accessible manuscript of their results is currently only available as a technical report on arXiv [24].

It is common in the development of semantics for non-monotonic reasoning, that authors provide counter-intuitive or artificial examples supposedly showing weaknesses of the discussed semantics. It is hardly possible to rule out the existence of such examples for the typicality model approach. While our approach is carefully designed to extend the materialisation-based approach, the satisfaction of the propositional KLM postulates as well as the discovery of strictly stronger postulates that capture the nature of DL reasoning and that are satisfied when reasoning with maximal typicality models is left for future work. As the study of DDLs is a very active research area, it also remains to put the typicality model approach into perspective with other recent non-monotonic extensions such as overriding [25], role-defeasibility [26] and context-based defeasibility [27]. Immediate extensions of our formalism could be achieved on the logic level, i.e. investigating more expressive DLs that also satisfy the canonical model property.

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